CS 280 Prelim 1 Solutions

For some of the solutions, we are only providing the answer. This is for your benefit: it is much faster for us to type up the answer than it is to show all of the steps. Thus, you get this solution set much faster.

1 Truth Tables

The most straightforward solution is $(\neg p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge r) \vee (p \wedge q \wedge \neg r)$.

2 Existential Quantification

(Note there are **many** *possible solutions to this one.)*

Let the domain for variable x be $\{1, 2\}$, $A(x)$ be the proposition "$x = 2$", $B(x)$ be the proposition "$x = 0$". Then $A(1)$ is false, and $B(1)$ is true, so $A(1) \rightarrow B(1)$ is true; therefore $\exists x[A(x) \rightarrow B(x)]$ is true.

However, $A(2)$ is true, and $\exists x B(x)$ is false. Therefore $\exists x A(x) \rightarrow \exists x B(x)$ is false.

3 Empty sets relating to sets

(a) $\varnothing \in \{\{\varnothing\}\}$ **No**
(b) $\varnothing \subseteq \{\{\varnothing\}\}$ **Yes**
(c) $\{\varnothing\} \in \{\{\varnothing\}\}$ **Yes**
(d) $\{\varnothing\} \subseteq \{\{\varnothing\}\}$ **No**– The subsets of $\{\{\varnothing\}\}$ are $\varnothing$ and $\{\{\varnothing\}\}$.
(e) $\{\varnothing\} \in P(\{\{\varnothing\}\})$ **No**– equivalent to (d).

4 1-1 functions

Let $f : S \longrightarrow T$ and $g : T \longrightarrow U$ be functions.

(a) **Claim:** If $g \circ f$ is 1-1, then f is 1-1.

Proof. Given $x, y \in S$, suppose $f(x) = f(y)$. Then $g(f(x)) = g(f(y))$ since g is a well-defined function, i.e. $(g \circ f)(x) = (g \circ f)(y)$. Since $(g \circ f)$ is 1-1, $x = y$. We conclude that f is also 1-1.

(b) *(Note there are* **many** *possible solutions to this one.)*

Let $S = T = U = R$, the set of real numbers. Define $g(x) = x^2$, and $f(x) = e^{x/2}$. $g(x)$ is not 1-1: for example, $g(1) = g(-1) = 1$. However, $(g \circ f)(x) = e^x$ is 1-1.

5 Matrices

The answer is $\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$.

$A \cdot B + A \cdot C + A \cdot D = A \cdot (B + C + D)$. And it is straight forward to see that $B + C + D$ is null.

6 Fermat's Little Theorem

$49 = 7^{1000002} \bmod 101$.

7 Chinese Remainder Theorem

Using the theorem, $x = 1002$.

8 Induction

Induction Hypothesis:

$$\sum_{k=1}^{n} \frac{1}{(2k-1)(2k+1)} = \frac{n}{2n+1}$$

Base case: $n = 1$. Clearly, $\frac{1}{(2-1)(2+1)} = \frac{1}{2+1}$.
Induction step: Assume the induction hypothesis is true for n. We prove it is true for $n+1$.

$$\sum_{k=1}^{n+1} \frac{1}{(2k-1)(2k+1)} = \sum_{k=1}^{n} \frac{1}{(2k-1)(2k+1)} + \frac{1}{(2(n+1)-1)(2(n+1)+1)}$$

$$= \frac{n}{2n+1} + \frac{1}{(2(n+1)-1)(2(n+1)+1)} \qquad \textit{by induction hypothesis}$$

$$= \frac{n}{2n+1} + \frac{1}{(2n+1)(2n+3)} = \frac{n(2n+3)}{(2n+1)(2n+3)} + \frac{1}{(2n+1)(2n+3)} = \frac{n(2n+3)+1}{(2n+1)(2n+3)}$$

$$= \frac{2n^2+3n+1}{(2n+1)(2n+3)} = \frac{(2n+1)(n+1)}{(2n+1)(2n+3)} = \frac{(n+1)}{2(n+1)+1}.$$

By induction, the hypothesis is true for any positive integer n.