CS 280, Final, December 16, 1999

Student's Name:

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Show all your work unless otherwise is stated. Each problem is 10 points worth.

1. a) Determine which of the following propositions are tautologies. Show why.

i) $(p \to \neg p) \to (p \to q)$ ii) $[(p \vee q) \to r] \leftrightarrow [(p \to r) \vee (q \to r)]$

b) Write down a proposition P over the variables p, q, r such that P is false only when p, q, r have truth values *false, true, false* or *true, false, false* respectively.

c) Can such an F be constructed using connectives $\wedge, \vee$ only? If yes, do. Otherwise show why not?

2. a) Let $L(x, y)$ be x *loves* y, b be Bob, m be Mary. Give English language translations of the following symbolic sentences (do not use variables in the answer)

i) $\forall x \exists y L(x, y)$

ii) $\exists x L(x, b) \to L(m, b)$

iii) $\forall x (L(x, m) \to \neg L(b, x))$

iv) $L(b, b) \to \neg L(b, m)$.

b) Which of the following sentences are logically equivalent (answers only)? In principle, you have to determine equivalence for all 6 pairs of sentences.

i) $\exists x (A(x) \to B(x))$, *ii)* $\exists x (A(x) \to \forall y A(y))$, *iii)* $\forall x A(x) \to \exists x B(x)$, *iv)* $\exists x B(x) \vee \forall y \neg B(y)$.

3. Which of the following are true (A and B are arbitrary sets)? Yes-No answers.

a) if $A \subseteq B$ and $B = \{\emptyset\}$ then $A = B$

b) if $A \subseteq B$ and $B = \{\emptyset\}$ then $A = \emptyset$

c) $\emptyset \in A$

d) $\emptyset \subseteq A$

e) $\overline{A \cup \overline{B}} = B \cap \overline{A}$

f) $A \cap B = \overline{(\overline{A} \cup \overline{B})}$

g) $(A - B) \cap (B - A) = \emptyset$

h) if $A \times B = \emptyset$ then $A = \emptyset$ and $B = \emptyset$

i) $A \subseteq A \times A$

j) $A \times A = A$ for some A

4. a) How many one-to-one functions from $\{a, b, c, d\}$ onto $\{1, 2, 3, 4, 5\}$ are there?

b) How many one-to-one functions f from $\{1, 2, 3, 4, 5\}$ onto itself are there such that $f = f^{-1}$?

c) Let f be a function. Which of the following are true? Yes-No answers.

i) $f(A \cap B) = f(A) \cap f(B)$

ii) $f(A \cup B) = f(A) \cup f(B)$

iii) $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$

iv) $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$

v) $f(\overline{A}) = \overline{f(A)}$

vi) $f^{-1}(\overline{C}) = \overline{f^{-1}(C)}$

5. Yes-No answers. Is it true that n^2 is $O(g(n))$, if $g(n)$ is

a) $100n^2 + 101n$

b) n^3

c) $n \log n$

d) $n^2 \log n$

e) $n^2 + \log n$

6. a) Use the Euclidean algorithm to find $\gcd(111, 2222)$
b) Find the inverse of 6 modulo 11
c) Find a positive integer x such that $x = 3 \pmod 9$ and $x = 5 \pmod{10}$

7. a) Perform the following operation on binary numbers

i) $11011 + 10110 =$

ii) $11011 \cdot 10110 =$

iii) $11011/10110 =$ (find a quotient and a remainder)

b) Transform $(11011)_2$ into decimal

c) How many bits can the binary expansion of a six decimal digit number have?

8. a) Give an example of matrices A and B such that $AB = \mathbf{0}$, but $A \neq \mathbf{0}$ and $B \neq \mathbf{0}$. Here $\mathbf{0}$ is the matrix having zero entries only.

b) Find all 2×2 zero-one matrices such that their boolean square is the identity matrix (an answer only).

c) How many additions and multiplications total does is take to multiply a 2×5 matrix by 4×6 matrix using the standard row-column algorithm?

9. a) Suppose that $S(n)$ be a proposition involving a nonnegative integer n, and that if $S(k)$ is false for all $k < n$ then so is $S(n)$. Which of the following are possible (answers only).

i) $S(n)$ is false for all $n \geq 0$

ii) There is an integer $M > 0$ such that $S(n)$ is true for all $n \leq M$ and false for all $n > M$

iii) $S(n)$ is true for all n

iv) There is an integer $M > 0$ such that $S(n)$ is false for all $n \leq M$ and true for all $n > M$.

b) A collection S of strings of characters is defined recursively by

- a belongs to S
- if Xa belongs to S then so do Xaa and Xb.

Which of the following belong to S:

i) b

ii) aa

iii) ab

iv) abb

v) aab

10. a) How many students must be in a class to guarantee that at least k students with the same birthday in the year 2000?

i) $k = 1$

ii) $k = 2$

iii) $k = 3$

b) A multiple choice exam has 20 questions each with 5 possible answers and 7 additional questions each with 3 possible answers. How many different answer sheets are possible?

11. a) In how many ways can a committee of 3 mathematicians and 5 computer scientists be selected from a panel of 20 having 10 mathematicians and 12 computer scientists?

b) Find the number of positive integer solutions of $x + y \leq 100$.

12. a) Find the probability that a family with six children has a boy and a girl (sexes of children are assumed equiprobable and independent).

b) What is the most likely number of boys?

c) Find the probability of having at least one girl given the first three children are boys.

13. A fair die is rolled until the sum of the spots exceeds 2. What is the expected number of rolls?

14. An integer is randomly selected from 1 to 100. What is the probability that it is divisible by 2 or by 3 but not by 5?

15. Give an example of a relation on $\{a, b, c\}$ which is

a) reflexive, not symmetric

b) irreflexive, symmetric

c) reflexive, symmetric, not transitive

16. a) Reorder the words in this phrase lexicographically.

b) Find the smallest possible poset having exactly two minimal elements and exactly three maximal elements. Does it have a greatest element and a least element (answers only)?

17. a) Indicate all possible vertices degrees occurring in the following graphs?

i) K_3

ii) C_{17}

iii) W_{10}

iv) $K_{3,5}$

v) Q_4

b) Is there a simple graph with exactly 3 vertices of degree 3?

c) Is there a simple graph with 10 vertices and 46 edges?

d) Is there a connected graph with 100 vertices, 96 of them of degree two and 4 of degree one?

18. Which of the following graphs have an Euler circuit? An Euler path? A Hamilton circuit? A Hamilton path? (answers only)

19. Which of the following graphs are planar? (answers only)

20. a) Represent $(p \to \neg p) \to (p \to q)$ using an ordered rooted tree.

b) What is the prefix form of the term $(x - y)^2 - (x + y)^2$?

c) What is the value of the prefix expression $- * 1/623$?