CS 280, Final, solutions December 16, 1999

1. a) i) tautology, ii) not, by truth tables
   b) $(p \vee \neg q \vee r) \wedge (\neg p \vee q \vee r)$
   c) No. Every proposition built from $\wedge, \vee$ is *false* when the atoms are all *false*.

2. a) i) Everybody loves somebody. ii) If somebody loves Bob then Mary does. iii) Bob does not love anyone who loves Mary. iv) If Bob loves himself then he does not love Mary.
   b) i $\sim$ iii, ii $\sim$ iv.

3. d, e, f, g, j.

4. a) None
   b)

$$1 + \binom{5}{2} + (1/2)\binom{5}{2}\binom{3}{2} = 26$$

c) i) NO. Let $f(x) = 1$ for all real x. Take $A = \{x \mid x < 0\}$, $B = \{x \mid x > 0\}$. Then $A \cap B = \emptyset$, therefore $f(A \cap B) = \emptyset$. On the other hand, $f(A) = f(B) = \{1\}$, thus $f(A) \cap f(B) = \{1\} \neq f(A \cap B)$.
ii) YES. $y \in f(A \cup B) \Leftrightarrow \exists x(x \in A \cup B \wedge y = f(x)) \Leftrightarrow \exists x((x \in A \vee x \in B) \wedge y = f(x)) \Leftrightarrow \exists x((x \in A \wedge y = f(x)) \vee (x \in B \wedge y = f(x))) \Leftrightarrow \exists x(x \in A \wedge y = f(x)) \vee \exists x(x \in B \wedge y = f(x)) \Leftrightarrow y \in f(A) \vee y \in f(B) \Leftrightarrow y \in f(A) \cup f(B)$.
iii) YES. $x \in f^{-1}(C \cap D) \Leftrightarrow f(x) \in (C \cap D) \Leftrightarrow [f(x) \in C \wedge f(x) \in D] \Leftrightarrow [x \in f^{-1}(C) \wedge x \in f^{-1}(D)] \Leftrightarrow x \in (f^{-1}(C) \cap f^{-1}(D))$.
iv) YES. (similar to iii).
v) NO. Take f and A as in (i). Then $f(\overline{A}) = \{1\}$, but $\overline{f(A)} = \mathbf{R} - \{1\}$.
vi) YES. $x \in f^{-1}(\overline{C}) \Leftrightarrow f(x) \in \overline{C} \Leftrightarrow \neg(f(x) \in C) \Leftrightarrow \neg(x \in f^{-1}(C)) \Leftrightarrow x \in \overline{f^{-1}(C)}$

5. a, b, d, e

6. $\gcd(111, 2222) = 1$; b) 2; c) 75

7. a) i) $11011 + 10110 = 110001 = (49)_{10}$, ii) $11011 \cdot 10110 = 1001010010 = (594)_{10}$, iii) $11011 = 1 \cdot 10110 + 101$
   b) $(11011)_2 = (27)_{10}$
   c) Any number from 17 to 20

8. a)

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

b)

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

c) None.

9. a) (i) is possible. Take $S(n)$ to be $n \neq n$.
(ii) and (iii) are impossible since $S(0)$ is necessarily false. Indeed, from the conditions it follows that $\forall k(k < 0 \rightarrow \neg S(k)) \rightarrow \neg S(0)$ holds. The antecedent $\forall k(k < 0 \rightarrow \neg S(k))$ is always true, since there are no $k < 0$ at all. Therefore, $\neg S(0)$ holds, meaning that $S(0)$ is false.
(iv) is impossible as it is clearly incompatible with the conditions.

b) i, ii, iii, v

10. a) i) 1, ii) 367, iii) 733 (by Pigeonhole Principle).
b) $3^7 \cdot 5^{20}$

11. a)

$$\binom{8}{1}\binom{10}{3} + 2\binom{8}{2}\binom{10}{4} + \binom{8}{3}\binom{10}{5}$$

b)

$$\binom{100}{2} = 4950$$

12. The sample space consists of 2^6 strings of B's and G's: *BBBBBB, BBBBBG, BBBBGB, ... ,GGGGGG*. Each of the outcomes is equally likely and thus has probability 1/64.
a) This event is supported by all outcomes other then *BBBBBB* and *GGGGGG*. The probability is 62/64=31/32.
b) The expected value of the number of boys is $E = 0 \cdot 1/64 + 1 \cdot 6/64 + 2 \cdot 15/64 + 3 \cdot 20/64 + 4 \cdot 15/64 + 5 \cdot 6/64 + 6 \cdot 1/64 = (6 + 30 + 60 + 60 + 30 + 6)/64 = 192/64 = 3$.
c) Use the formula $P(A \mid B) = P(A \cap B)/P(B)$. Let A be event "at least one girl," and B be event "the first three are boys." Then $P(B) = 8/64$, $P(A \cap B) = 7/64$, therefore $P(A \mid B) = 7/8$.

13. By the formula for expected value, 3(1/36)+2(11/36)+1(2/3)=49/36

14. Let A be event "divisible by 2," B - "divisible by 3" and C - "not divisible by 5." We have to find the probability of $(A \cup B) \cap C$, which is equal to $(A \cap C) \cup (B \cap C)$. By the inclusion-exclusion formula, the desired probability $P((A \cap C) \cup (B \cap C)) = P(A \cap C) + P(B \cap C) - P(A \cap B \cap C)$. Now, $P(A \cap C) = 40/100$ (there are 40 integers divisible by 2 and not divisible by 5), $P(B \cap C) = 27/100$, $P(A \cap B \cap C) = 13/100$. Thus the desired probability is equal to $40/100 + 27/100 - 13/100 = 54/100$.

15. a) $\{(a,a),\ (b,b),\ (c,c),\ (a,b)\}$
b) $\emptyset$
c) $\{(a,a),\ (b,b),\ (c,c),\ (a,b),\ (b,a),\ (b,c),\ (c,b)\}$

16. a) in, lexicographically, phrase, Reorder, the, this, words
b) $S = \{a,b,c,d\},\ b \preceq c,\ b \preceq d$. No, it does not.

17. a) i) 2; ii) 2; iii) 3,10; iv) 3,5; v) 4

b) Yes.
c) No.
d) No.

18. i) HC, HP; ii) EP, HC, HP; iii) HC, HP; iv) None; v) HP

19. i, ii, iv

20. a) trivial
b) Represent $(x-y) \uparrow 2-(x+y) \uparrow 2$ as a binary tree, and then read this tree in the preorder: $- \uparrow -xy2 \uparrow +xy2$
c) Convert this prefix expression to the standard one and then perform the computations: $1 * (6/2) - 3 = 1 \cdot 3 - 3 = 0$.