Euler Paths, Planar Graphs and Hamiltonian Paths

Some Graph Theory Terms

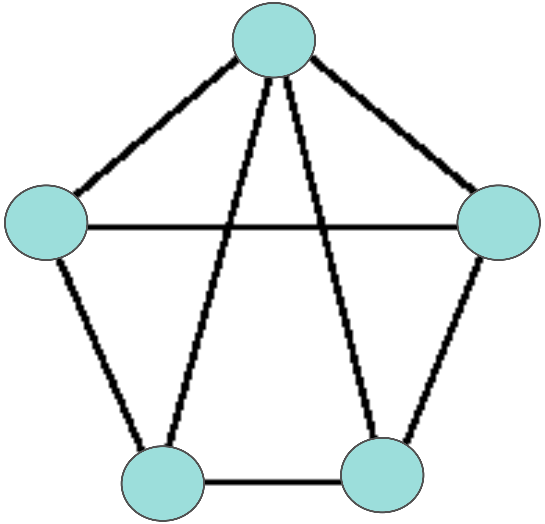
- Degree of node A
  - The number of edges that include A
- Strongly Connected Component
  - A set of nodes where there is an path between any two nodes in the set
- Bridge
  - An edge between nodes in a strongly connected component such that, if the edge was removed, the nodes are no longly a strongly connected component

Graph Coloring

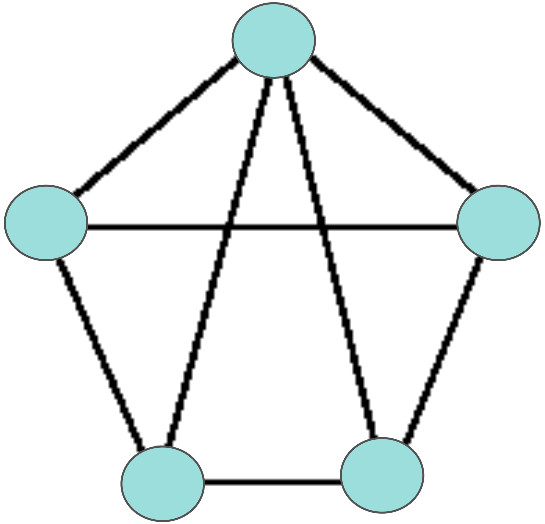
- Assignment of colors to the vertices of a graph such that no two adjacent vertices have the same color
- If a graph is n-colorable it means that using at most n colors the graph can be colored such that adjacent vertices don’t have the same color
- Chromatic number is the smallest number of colors needed to color a graph
- In general, computing the chromatic number of a graph is a hard problem

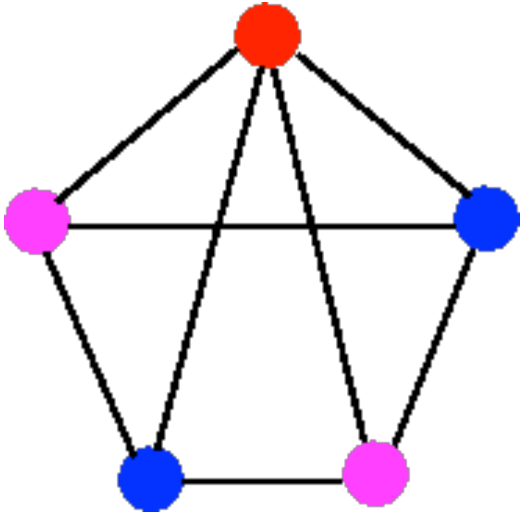
Graph Coloring Example

How many colors do you need to color this graph?

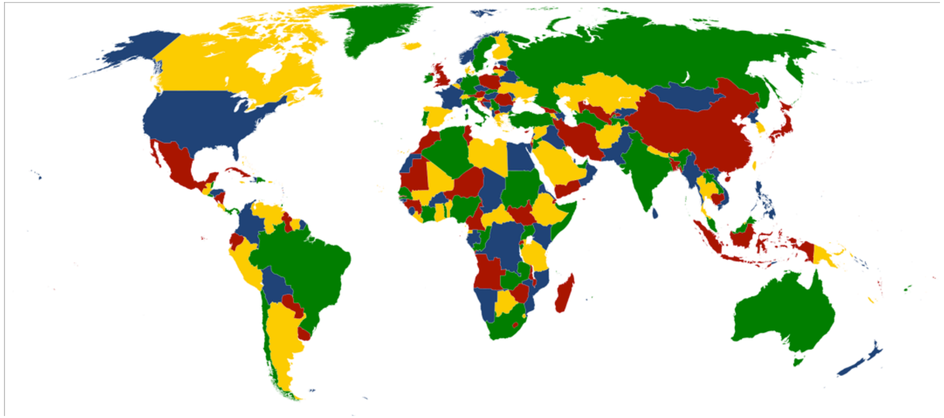


Graph Coloring Example





Planar Graphs



- A graph that can be drawn such that no edges overlap
- A key example of planar graphs is a map where every country is a node and the edges represent having shared borders

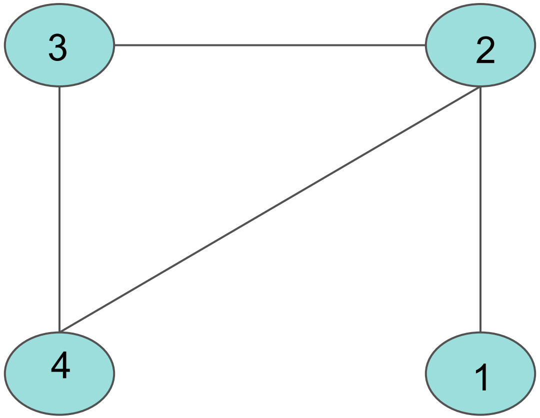
Four Color Theorem

- Every planar graph is 4 colorable
- Proposed in the 1800’s
- First proven in 1976 with a computer proof assistant
  - The proof was considered controversial at the time
  - Now more modern and simplified version are generally accepted

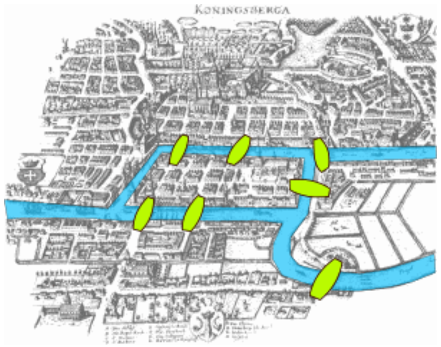
Euler Paths

- Path which uses every edge exactly once
- An undirected graph has an Eulerian path if and only if exactly zero or two vertices have odd degree

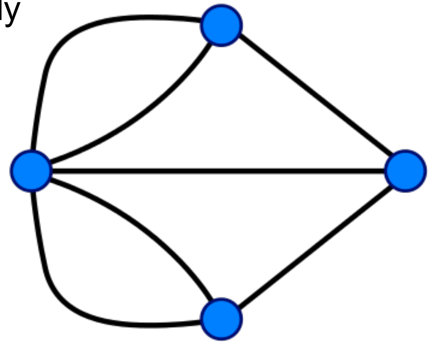
Euler Path Example



History of the Problem/Seven Bridges of Königsberg



- Is there a way to map a tour through Königsberg crossing every bridge exactly once
- Famous mathematician Leonhard Euler proved not only that it was impossible for this city, but generalized It and laid the foundations of graph theory



How to Find an Eulerian Path

- Select a starting node
  - If all nodes are of even degree, any node works
  - If there are two odd degree nodes, pick one of them
- While the current node has remaining edges
  - Choose an edge, if possible pick one that is not a bridge
  - Set the current node to be the node across that edge
  - Delete the edge from the graph

Hamiltonian Paths

- Path which visits every vertex exactly once
- Very hard to determine if a graph has a Hamiltonian path
- However, if you given a path, it is easy and efficient to verify if it is a Hamiltonian Path

P and NP Problems

- P
  - Polynomial Time
  - The set of problems that can be solved in time polynomial in the size of the input
- NP
  - Nondeterministic Polynomial Time
  - The set of problems where a potential solution (possibly alongside some supporting evidence) can be verified in time polynomial in the size of the input

Reductions

- Suppose we have two problems A and B as well as an efficient algorithm for turning a statement of an A problem into a B problem and an answer to a B problem into an A problem
- Then this gives us an efficient solution to A provided we have an efficient solution to B
- In other words, A is at most as hard as B

NP Complete

- The set of problems that are both at most or at least as hard as any problem in NP
- Proving a problem is NP Complete requires proving it is in NP and that it is at least as hard as the specific problem 3SAT
  - This means you need to find a reduction from 3SAT to the problem

3SAT

- Suppose we have a sequence of logical variables x1,x2,...xn
- Clause
  - 3 signed variables
  - xi or -xi
  - Taken as a logical or
- Suppose we have a sequence of clauses over those variables
- Does there exist an assignment from these variables to truth values such that each clause is satisfied

Clique

- Suppose we have a graph and a positive number k
- Does there exist a set of k nodes such that there is an edge between each node in the set?

Reduction: 3SAT -> Clique

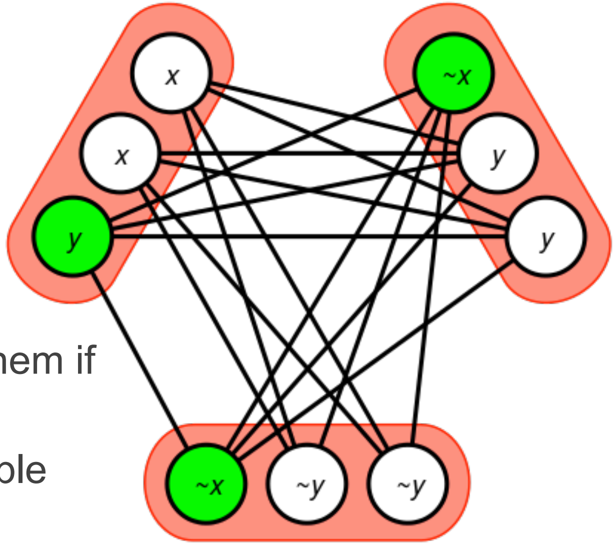
- For each clause create 3 nodes

  Each labelled with the variable name and sign

- Given two nodes, construct an edge between them if
  - They are from different clauses
  - If they correspond to the same variable

    then they have the same sign

- Does this graph have a k clique, where k is the number of clauses



Examples of NP Complete Problems

- 3SAT
- Clique
- Vertex Cover
- Hamiltonian Path
- Travelling Salesman
- Graph Coloring
- Many of these have useful applications