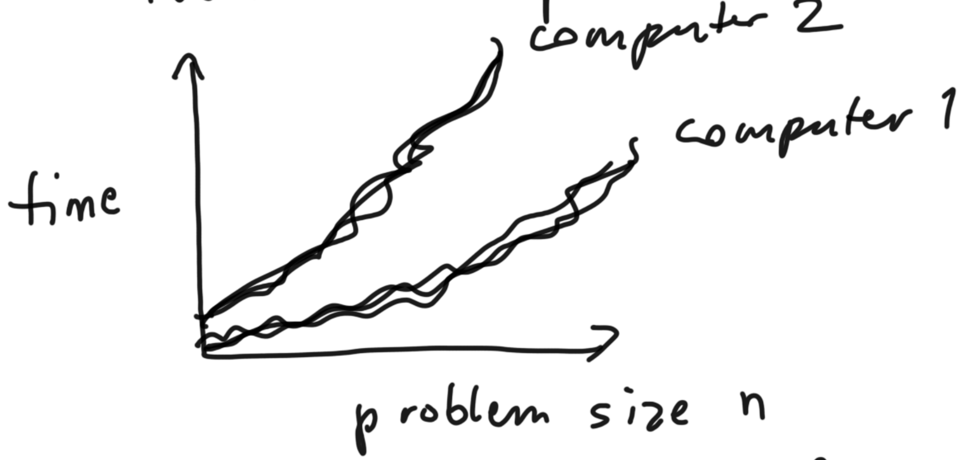
CS 2110 - Asymptotic Complexity

"good" algorithms

- Hard to compare algorithms across computers, input



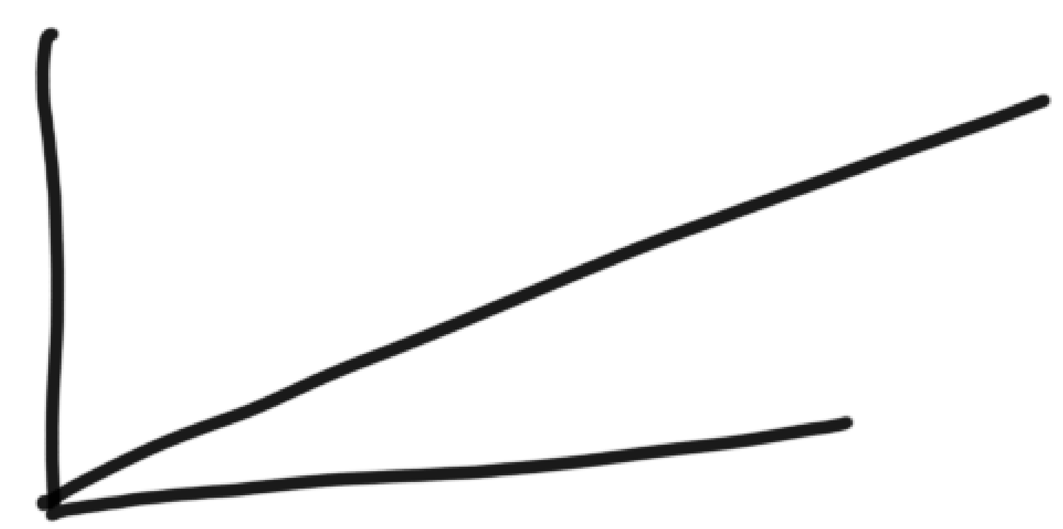
Machine-independent performance?

1. zoom out
   to large n
2. use worst-case
   or average-case
   performance
3. ignore constant factors

Big-O notation

Describes "big picture" behavior of functions at large n
(asymptotic complexity)

$O(n)$: functions whose behavior is linear for large n



$O(n)$ = set of functions, e.g. n, $2n+1$, $1000000n$

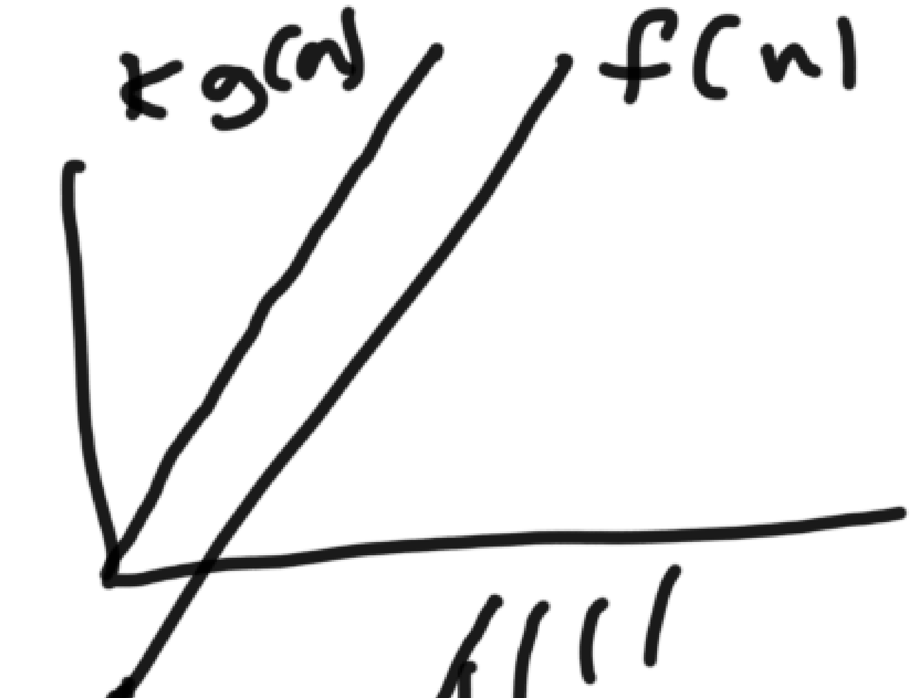
not: n^2

$f(n)$ is $O(g(n))$ means $f(n) \in O(g(n))$

or: $f(n) = O(g(n))$ (shorthand)

<u>Defn</u>

$f(n)$ is $O(g(n))$ if there exists constant multiplier k and some value n_0 such that $f(n) \le k \cdot g(n)$ for all $n \ge n_0$



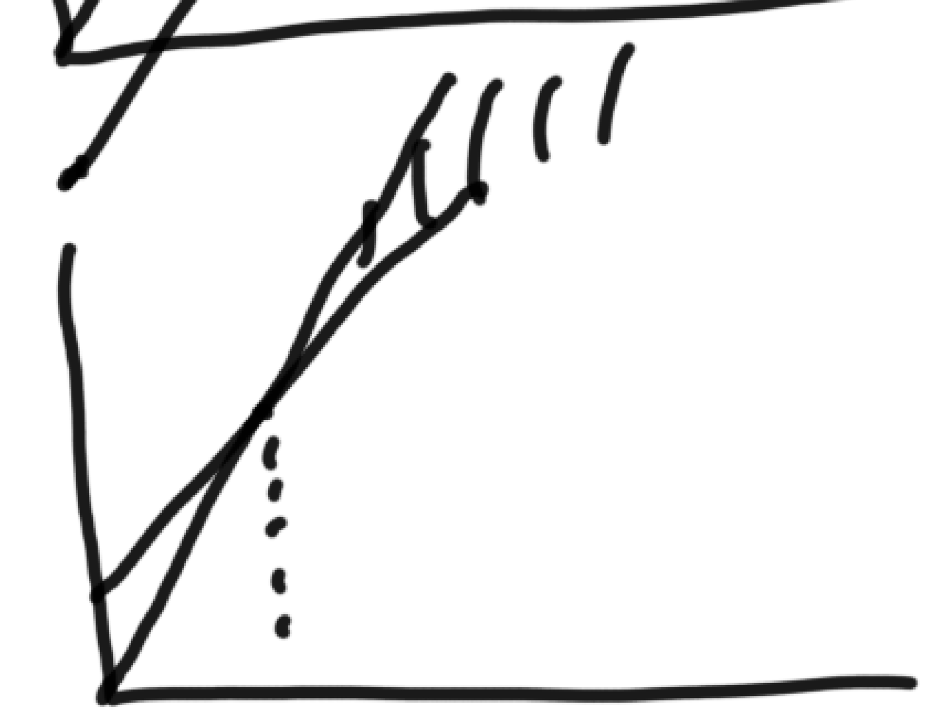
Ex/ $3n-2$ is $O(n)$?

<u>witness</u> $k=3$, $n_0=0$

$3n+2$ is $O(n)$?

$k=4$, $n_0=2$ $\quad 3n+2 \le 4n$

$n \ge 2$



$n_0=2$

Is $3n+2$ $\underline{\underline{O(n^2)}}$?

$\boxed{A: \text{Yes}}$

B: No

$k = 4$, $n_0 = 2$ $\qquad 4n^2 \geq 4n \geq 3n+2$

for $n \geq 2$

$O(g(n))$ is an <u>upper bound</u>

n^2 is $O(n)$? No.

$n^2 > kn$ for $n > k$

$O(1)$: constant time (max time c_1)

$O(n)$: linear time

$O(n^2)$: quadratic time

$O(n^3)$: cubic time

$O(n^k)$: polynomial time

$O(k^n)$: exponential time

$O(\lg n)$: logarithmic

Combining asymptotic complexity

Any constant c is $O(1)$.

$O(c \cdot f(n)) = O(f(n))$

$O(n^x) \subseteq O(n^y)$ if $x \leq y$

$O(n^x) \subset O(n^y)$ if $x < y$

If f_1 is $O(g_1(n))$ & f_2 is $O(g_2(n))$ $\Rightarrow$ $\boxed{f_1(n) + f_2(n)}$ is $O(g_1(n) + g_2(n))$

(pick larger k)

pun: $\underline{O(g_1(n)) + O(g_2(n))} = O(g_1(n) + g_2(n))$

set of functions constructed by choosing one func ~~fcs~~ from each side

$$O(f)\cdot O(g) = O(f\cdot g)$$

$$O(n)\cdot O(n) = O(n^2)$$

base 2

$$\log_c n = \frac{\log n}{\log c} = O(\log n) = O(\lg n)$$

But: $O(a^n) \neq O(b^n)$ if $a \neq b$

Eg. $\frac{3^n}{2^n}$ grows to ∞

2^n is $O(3^n)$

3^n is not $O(2^n)$

$$O(n^2) + O(n^2) + O(n^2)$$

$$n^2 + \cancel{2n} + \cancel{1000}$$

A: $O(1)$

B: $O(n)$

$\boxed{C: O(n^2)}$

D: NONE OF ABOVE

E: $O(\lg n)$

Using ratios

if $g(n) > 0$ $\quad f(n) \le k \cdot g(n) \iff \dfrac{f(n)}{g(n)} \le k$

So $\displaystyle\lim_{n\to\infty} \frac{f(n)}{g(n)} \le k \implies f(n)$ is $O(g(n))$

$\displaystyle\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty \implies f(n)$ is not $O(g(n))$

Ex/ $n \lg n$ is $O(n)$?

$\displaystyle\lim_{n\to\infty} \frac{n \lg n}{n} = \lim_{n\to\infty} \lg n = \infty$

NO

L'Hôpital's Rule

If $\lim_{n\to\infty} f(n) = \lim_{n\to\infty} g(n) = \infty$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \boxed{\lim_{n\to\infty} \frac{f'(n)}{g'(n)}}$$

Ex/ $\ln n \in O(n)$? YES

$$\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{n\to\infty} \frac{(1/n)}{1} = \lim_{n\to\infty} \frac{1}{n} = 0$$

$\ln^2 n$ is $O(n)$?

$$\lim \frac{\ln^2 n}{n} = \lim \frac{2\ln n \cdot \frac{1}{n}}{1}$$
$$= \lim \frac{\ln n'}{n} = 0$$

$\ln^k n$ is $O(n^\varepsilon)$ ~~for~~ for any $\varepsilon > 0$.