SORTING AND ASYMPTOTIC COMPLEXITY

Lecture 14
CS2110 – Fall 2013

Reading and Homework

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- Texbook, chapter 8 (general concepts) and 9 (MergeSort, QuickSort)
- Thought question: Cloud computing systems sometimes sort data sets with hundreds of billions of items – far too much to fit in any one computer. So they use multiple computers to sort the data. Suppose you had N computers and each has room for D items, and you have a data set with N*D/2 items to sort. How could you sort the data? Assume the data is initially in a big file, and you'll need to read the file, sort the data, then write a new file in sorted order.

InsertionSort

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```
//sort a[], an array of int
for (int i = 1; i < a.length; i++) {
  // Push a[i] down to its sorted position
  //         in a[0..i]
  int temp = a[i];
  int k;
  for (k = i;  0 < k && temp < a[k–1];  k– –)
        a[k] = a[k–1];
   a[k] = temp;
}
```

- Worst-case: O(n^2) (reverse-sorted input)
- Best-case: O(n) (sorted input)
- Expected case: O(n^2)
- Expected number of inversions: n(n–1)/4

- Many people sort cards this way
- Invariant of main loop: ***a[0..i-1] is sorted***
- Works especially well when input is *nearly sorted*

SelectionSort

```
//sort a[], an array of int
for (int i = 1; i < a.length; i++) {
  int m= index of minimum of a[i..];
  Swap b[i] and b[m];
}
```

Another common way for people to sort cards

Runtime
- Worst-case O(n^2)
- Best-case O(n^2)
- Expected-case O(n^2)

	0	i	length
a	sorted, smaller values	larger values	

Each iteration, swap min value of this section into a[i]

Divide & Conquer?

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It often pays to
- Break the problem into smaller subproblems,
- Solve the subproblems separately, and then
- Assemble a final solution

This technique is called *divide-and-conquer*
- Caveat: It won't help unless the *partitioning* and *assembly* processes are inexpensive

Can we apply this approach to sorting?

MergeSort

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- Quintessential divide-and-conquer algorithm
- Divide array into equal parts, sort each part, then merge
- Questions:
  - Q1: How do we divide array into two equal parts?
    A1: Find middle index: a.length/2
  - Q2: How do we sort the parts?
    A2: Call MergeSort recursively!
  - Q3: How do we merge the sorted subarrays?
    A3: Write some (easy) code

Merging Sorted Arrays A and B into C

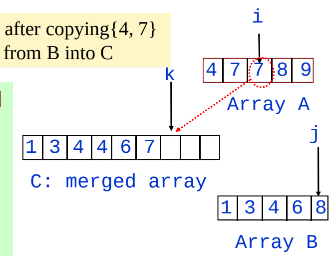
Picture shows situation after copying{4, 7} from A and {1, 3, 4, 6} from B into C

A[0..i-1] and B[0..j-1] have been copied into C[0..k-1].

C[0..k-1] is sorted.

Next, put a[i] in c[k], because a[i] < b[j].

Then increase k and i.



Merging Sorted Arrays **A** and **B** `into C`

- Create array C of size: size of A + size of B
- i= 0; j= 0; k= 0; // initially, nothing copied
- Copy smaller of A[i] and B[j] into C[k]
- Increment i or j, whichever one was used, and k
- When either A or B becomes empty, copy remaining elements from the other array (B or A, respectively) into C

This tells what has been done so far:

A[0..i-1] and B[0..j-1] have been placed in C[0..k-1].

C[0..k-1] is sorted.

`MergeSort` Analysis

Outline (code on website)

- Split array into two halves
- Recursively sort each half
- Merge two halves

- Merge: combine two sorted arrays into one sorted array
  - Rule: always choose smallest item
  - Time: O(n) where n is the total size of the two arrays

Runtime recurrence

T(n): time to sort array of size n
 T(1) = 1
 T(n) = 2T(n/2) + O(n)

Can show by induction that
 T(n) is O(n log n)

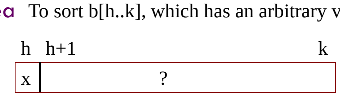
Alternatively, can see that T(n) is O(n log n) by looking at tree of recursive calls

`MergeSort` Notes

- Asymptotic complexity: O(n log n)
  Much faster than $O(n^2)$
- Disadvantage
  - Need extra storage for temporary arrays
  - In practice, can be a disadvantage, even though MergeSort is *asymptotically optimal for sorting*
  - Can do MergeSort in place, but *very* tricky (and slows execution significantly)
- Good sorting algorithms that do not use so much extra storage?
  Yes: QuickSort

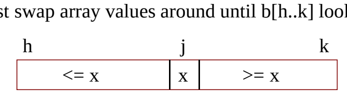
`QuickSort`

Idea To sort b[h..k], which has an arbitrary value x in b[h]:



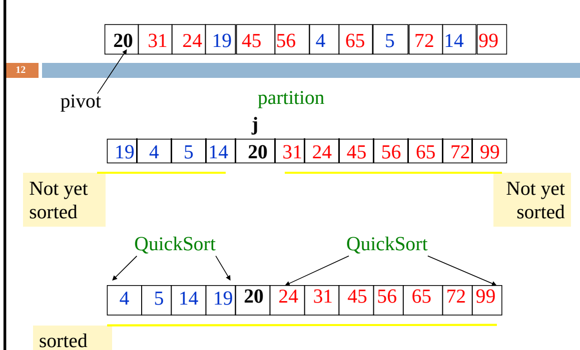
x is called the pivot

first swap array values around until b[h..k] looks like this:



Then sort b[h..j-1] and b[j+1..k] —how do you do that?

Recursively!



In-Place Partitioning

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- On the previous slide we just moved the items to partition them
- But in fact this would require an extra array to copy them into
- Developer of QuickSort came up with a better idea
  - *In place partitioning* cleverly splits the data ***in place***

In-Place Partitioning

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Key issues

- How to choose a *pivot*?
- How to *partition* array in place?

Partitioning in place

- Takes O(n) time (next slide)
- Requires no extra space

Choosing pivot

- Ideal pivot: the median, since it splits array in half
- Computing median of unsorted array is O(n), quite complicated

Popular heuristics: Use

- first array value (not good)
- middle array value
- median of first, middle, last, values GOOD!
- Choose a random element

In-Place Partitioning

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Change b[h..k] from this:

```
     h  h+1                       k
b  | x |           ?              |
```

to this by repeatedly swapping array elements:

```
     h                 j          k
b  |     <= x       | x |  >= x   |
```

Do it one swap at a time, keeping the array looking like this. At each step, swap b[j+1] with something

```
     h          j        t        k
b  |  <= x    | x |  ?   |  >= x  |
```

Start with: j= h; t= k;

In-Place Partitioning

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```
     h          j      t       k
b  |  <= x    | x | ?  | >= x  |
```

Initially, with j = h and t = k, this diagram looks like the start diagram

```
j= h; t= k;
while (j < t) {
   if (b[j+1] <= x) {
      Swap b[j+1] and b[j];   j= j+1;
   } else {
      Swap b[j+1] and b[t];   t= t-1;
   }
}
```

Terminates when j = t, so the "?" segment is empty, so diagram looks like result diagram

In-Place Partitioning

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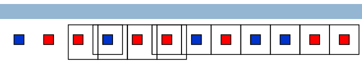
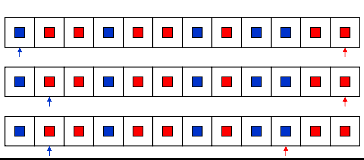


How can we move all the blues to the left of all the reds?

- Keep two indices, LEFT and RIGHT
- Initialize LEFT at start of array and RIGHT at end of array
  Invariant: all elements to left of LEFT are blue
  all elements to right of RIGHT are red
- Keep advancing indices until they pass, maintaining invariant



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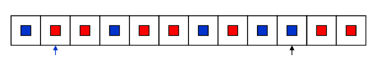
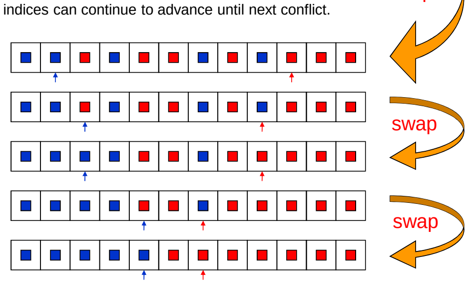
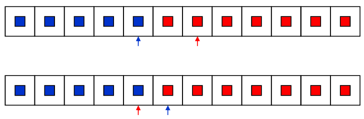


Now neither LEFT nor RIGHT can advance and maintain invariant.
We can swap red and blue pointed to by LEFT and RIGHT indices.
After swap, indices can continue to advance until next conflict.





- Once indices cross, partitioning is done
- If you replace blue with ≤ **p** and red with ≥ **p**, this is exactly what we need for QuickSort partitioning
- Notice that after partitioning, array is partially sorted
- Recursive calls on partitioned subarrays will sort subarrays
- No need to copy/move arrays, since we partitioned in place

QuickSort procedure

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```
/** Sort b[h..k]. */
public static void QS(int[] b, int h, int k) {
   if (b[h..k] has < 2 elements) return;      Base case
   int j= partition(b, h, k);
   // We know b[h..j–1] <= b[j] <= b[j+1..k]
   // So we need to sort b[h..j-1] and b[j+1..k]
   QS(b, h, j-1);
   QS(b, j+1, k);
}
```

Function does the partition algorithm and returns position j of pivot

QuickSort versus MergeSort

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```
/** Sort b[h..k] */
public static void QS
      (int[] b, int h, int k) {
   if (k – h < 1) return;
   int j= partition(b, h, k);
   QS(b, h, j-1);
   QS(b, j+1, k);
}
```

```
/** Sort b[h..k] */
public static void MS
      (int[] b, int h, int k) {
   if (k  – h < 1) return;
   MS(b, h, (h+k)/2);
   MS(b, (h+k)/2 + 1, k);
   merge(b, h, (h+k)/2, k);
}
```

One processes the array then recurses.
One recurses then processes the array.

QuickSort Analysis

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Runtime analysis (worst-case)

- Partition can produce this: | p | ≥ p |
- Runtime recurrence: $T(n) = T(n–1) + n$
- Can be solved to show worst-case $T(n)$ is $O(n^2)$
- Space can be $O(n)$ —max depth of recursion

Runtime analysis (expected-case)

- More complex recurrence
- Can be solved to show *expected* $T(n)$ is $O(n \log n)$

Improve constant factor by avoiding QuickSort on *small* sets

- Use InsertionSort (for example) for sets of size, say, ≤ 9
- Definition of *small* depends on language, machine, etc.

Sorting Algorithm Summary

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We discussed

- InsertionSort
- SelectionSort
- MergeSort
- QuickSort

Other sorting algorithms

- HeapSort (will revisit)
- ShellSort (in text)
- BubbleSort (nice name)
- RadixSort
- BinSort
- CountingSort

Why so many? Do computer scientists have some kind of sorting fetish or what?

Stable sorts: **Ins, Sel, Mer**

Worst-case O(n log n): **Mer, Hea**

Expected O(n log n):
Mer, Hea, Qui

Best for nearly-sorted sets: **Ins**

No extra space: **Ins, Sel, Hea**

Fastest in practice: **Qui**

Least data movement: **Sel**

Lower Bound for Comparison Sorting

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Goal: Determine minimum time *required* to sort *n* items

Note: we want *worst-case*, not *best-case* time

- Best-case doesn't tell us much. E.g. Insertion Sort takes O(n) time on already-sorted input
- Want to know *worst-case time* for *best possible* algorithm

- How can we prove anything about the *best possible* algorithm?
- Want to find characteristics that are common to *all* sorting algorithms
- Limit attention to *comparison-based algorithms* and try to count number of comparisons

Comparison Trees

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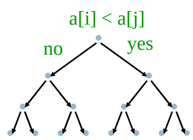
- Comparison-based algorithms make decisions based on comparison of data elements
- Gives a *comparison tree*
- If algorithm fails to terminate for some input, comparison tree is infinite
- Height of comparison tree represents *worst-case number of comparisons* for that algorithm
- Can show: *Any* correct comparison-based algorithm must make at least n log n comparisons in the worst case



Lower Bound for Comparison Sorting

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- Say we have a correct comparison-based algorithm
- Suppose we want to sort the elements in an array b[]
- Assume the elements of b[] are distinct
- Any permutation of the elements is initially possible
- When done, b[] is sorted
- But the algorithm could not have taken the same path in the comparison tree on different input permutations

Lower Bound for Comparison Sorting

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How many input permutations are possible? n! ~ $2^{n \log n}$

For a comparison-based sorting algorithm to be correct, it must have at least that many leaves in its comparison tree

To have at least n! ~ $2^{n \log n}$ leaves, it must have height at least n log n (since it is only binary branching, the number of nodes at most doubles at every depth)

Therefore its longest path must be of length at least n log n, and that it its worst-case running time

`java.lang.Comparable<T>` Interface

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`public int compareTo(T x);`

- Return a negative, zero, or positive value
  - negative if **this** is before **x**
  - 0 if **this.equals(x)**
  - positive if **this** is after **x**

Many classes implement **`Comparable`**

- **`String`**, **`Double`**, **`Integer`**, **`Character`**, **`Date`**, …
- Class implements Comparable? Its method compareTo is considered to define that class's *natural ordering*

Comparison-based sorting methods should work with **`Comparable`** for maximum generality