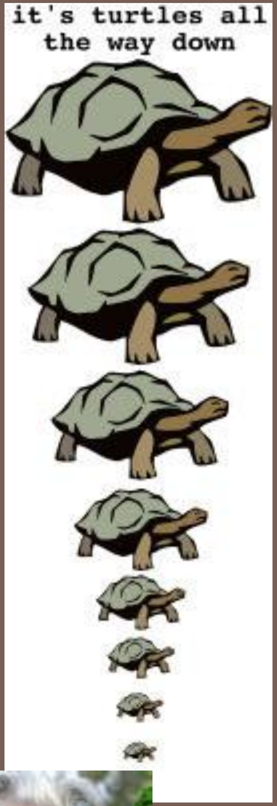



***A well-known scientist (some say it was Bertrand Russell) once gave a public lecture on astronomy. He described how the earth orbits around the sun and how the sun, in turn, orbits around the center of a vast collection of stars called our galaxy.***

***At the end of the lecture, a little old lady at the back of the room got up and said: "What you have told us is rubbish. The world is really a flat plate supported on the back of a giant tortoise." The scientist gave a superior smile before replying, "What is the tortoise standing on?" "You're very clever, young man, very clever", said the old lady. "But it's turtles all the way down!"***

# INDUCTION

Lecture 22

CS2110 – Fall 2010

# Overview: Reasoning about programs



- Our broad problem: code is unlikely to be correct if we don't have good reasons for believing it works
  - We need clear problem statements
  - And then a rigorous way to convince ourselves that what we wrote solves the problem
- But reasoning about programs can be hard
  - Especially with recursion, concurrency
  - Today focus on recursion

# Overview: Reasoning about programs



- Recursion
  - A programming strategy that solves a problem by reducing it to simpler or smaller instance(s) of the same problem
- Induction
  - A mathematical strategy for proving statements about natural numbers 0,1,2,... (or more generally, about inductively defined objects)
- They are very closely related

- Induction can be used to establish the *correctness* and *complexity* of programs

# Defining Functions



- It is often useful to describe a function in different ways
  - Let $S : \text{int} \rightarrow \text{int}$ be the function where S(n) is the sum of the integers from 0 to n. For example,

    $S(0) = 0$ $\qquad$ $S(3) = 0+1+2+3 = 6$

  - Definition: iterative form
    - $S(n) = 0+1+ \ldots + n$

      $= \sum_{i=0}^{n} i$

  - Another characterization: closed form
    - $S(n) = n(n+1)/2$

# Sum of Squares



- A more complex example
  - Let SQ : int → int be the function that gives the sum of the squares of integers from 0 to n:

    $SQ(0) = 0$

    $SQ(3) = 0^2 + 1^2 + 2^2 + 3^2 = 14$

- Definition (iterative form):

  $SQ(n) = 0^2 + 1^2 + \dots + n^2$

- Is there an equivalent closed-form expression?

# Closed-Form Expression for SQ(n)



- Sum of integers between 0 through n was $n(n+1)/2$ which is a *quadratic* in n (that is, $O(n^2)$)

- Inspired guess: perhaps sum of *squares* of integers between 0 through n is a *cubic* in n

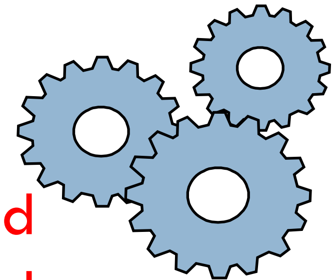
- Conjecture: $SQ(n) = an^3+bn^2+cn+d$
  where a, b, c, d are unknown coefficients

- How can we find the values of the four unknowns?
  - Idea: Use any 4 values of n to generate 4 linear equations, and then solve

# Finding Coefficients



$$SQ(n) = 0^2+1^2+\ldots+n^2 = an^3+bn^2+cn+d$$



- Use n = 0, 1, 2, 3
  - $SQ(0) = 0 = a\cdot 0 + b\cdot 0 + c\cdot 0 + d$
  - $SQ(1) = 1 = a\cdot 1 + b\cdot 1 + c\cdot 1 + d$
  - $SQ(2) = 5 = a\cdot 8 + b\cdot 4 + c\cdot 2 + d$
  - $SQ(3) = 14 = a\cdot 27 + b\cdot 9 + c\cdot 3 + d$

  - Solve these 4 equations to get
    - $a = 1/3 \quad b = 1/2 \quad c = 1/6 \quad d = 0$

# Is the Formula Correct?



- This suggests

$$SQ(n) = 0^2 + 1^2 + \ldots + n^2$$
$$= n^3/3 + n^2/2 + n/6$$
$$= n(n+1)(2n+1)/6$$

- Question: Is this closed-form solution true for all n?
  - Remember, we only used n = 0,1,2,3 to determine these coefficients
  - We do not know that the closed-form expression is valid for other values of n

# One Approach



- Try a few other values of n to see if they work.
  - Try n = 5: SQ(n) = 0+1+4+9+16+25 = 55
  - Closed-form expression: $5 \cdot 6 \cdot 11/6 = 55$
  - Works!

- Try some more values…

- We can never prove validity of the closed-form solution for all values of n this way, since there are an infinite number of values of n

# A Recursive Definition



- To solve this problem, let's express SQ(n) in a different way:
  - $SQ(n) = 0^2 + 1^2 + \ldots + (n-1)^2 + n^2$
  - The part in the box is just $SQ(n-1) + n^2$
- This leads to the following *recursive* definition
  - $SQ(0) = 0$ ← Base Case
  - $SQ(n) = SQ(n-1) + n^2, \; n > 0$ ← Recursive Case
- Thus,
  - $SQ(4) = SQ(3) + 4^2 = SQ(2) + 3^2 + 4^2 = SQ(1) + 2^2 + 3^2 + 4^2 = SQ(0) + 1^2 + 2^2 + 3^2 + 4^2 = 0 + 1^2 + 2^2 + 3^2 + 4^2$

# Are These Two Functions Equal?



- $SQ_r$ (r = recursive)

$$SQ_r(0) = 0$$

$$SQ_r(n) = SQ_r(n-1) + n^2, \quad n > 0$$

- $SQ_c$ (c = closed-form)

$$SQ_c(n) = n(n+1)(2n+1)/6$$

# Induction over Integers



- To prove that some property P(n) holds for all integers $n \geq 0$,

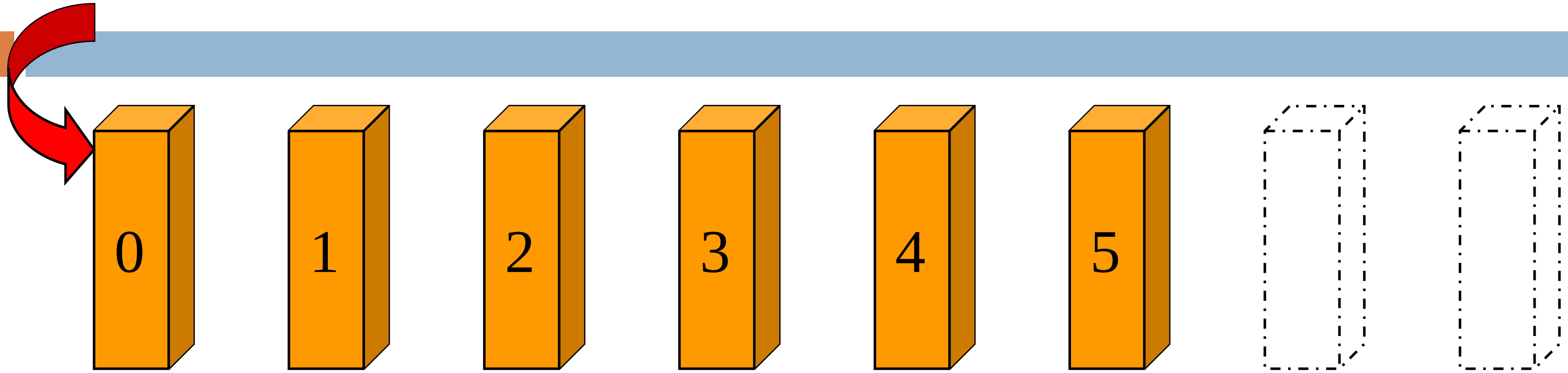
1. **Basis:** Show that P(0) is true

2. **Induction Step:** Assuming that P(k) is true for an *unspecified* integer k, show that P(k+1) is true

- **Conclusion:** Because we could have picked any k, we conclude that P(n) holds for all integers $n \geq 0$

# Dominos






- Assume equally spaced dominos, and assume that spacing between dominos is less than domino length
- How would you argue that all dominos would fall?
- Dumb argument:
  - Domino 0 falls because we push it over
  - Domino 0 hits domino 1, therefore domino 1 falls
  - Domino 1 hits domino 2, therefore domino 2 falls
  - Domino 2 hits domino 3, therefore domino 3 falls
  - …
- Is there a more compact argument we can make?

# Better Argument

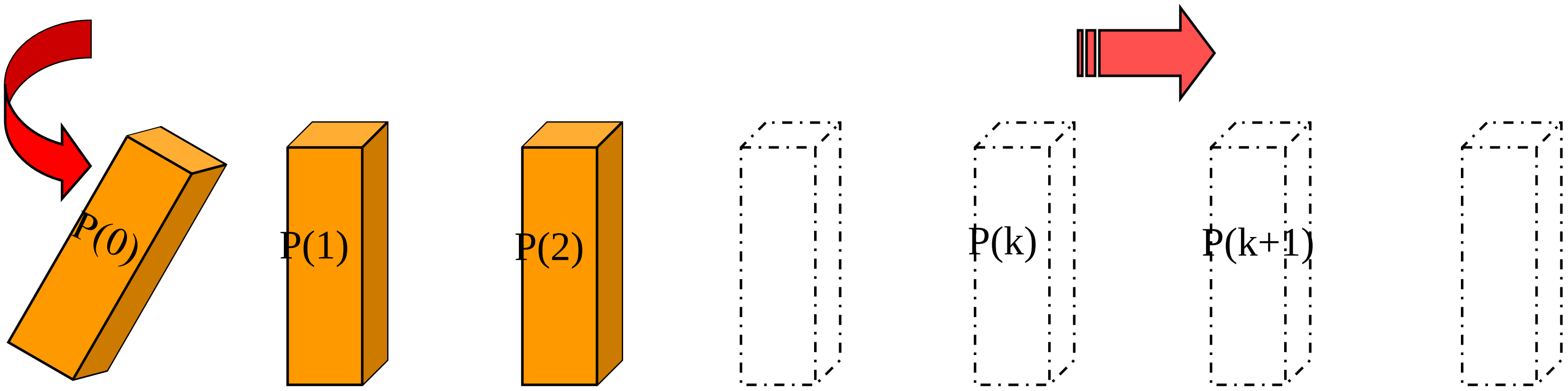


- Argument:
  - Domino 0 falls because we push it over (Base Case or Basis)
  - Assume that domino k falls over (Induction Hypothesis)
  - Because domino k's length is larger than inter-domino spacing, it will knock over domino k+1 (Inductive Step)
  - Because we could have picked any domino to be the $k^{th}$ one, we conclude that all dominos will fall over (Conclusion)
- This is an inductive argument
- This version is called *weak induction*
  - There is also *strong induction* (later)
- Not only is this argument more compact, it works for an arbitrary number of dominoes!

# $SQ_r(n) = SQ_c(n)$ for all n?



- Define P(n) as $SQ_r(n) = SQ_c(n)$




- Prove P(0)
- Assume P(k) for unspecified k, and then prove P(k+1) under this assumption

# Proof (by Induction)



- Recall: $SQ_r(0) = 0$
  $SQ_r(n) = SQ_r(n-1) + n^2, \quad n > 0$

  $SQ_c(n) = n(n+1)(2n+1)/6$
- Let P(n) be the proposition that $SQ_r(n) = SQ_c(n)$
- Basis: P(0) holds because $SQ_r(0) = 0$ and $SQ_c(0) = 0$ by definition
- Induction Hypothesis: Assume $SQ_r(k) = SQ_c(k)$
- Inductive Step:
  $SQ_r(k+1) \quad = SQ_r(k) + (k+1)^2$ by definition of $SQ_r(k+1)$ =
  $SQ_c(k) + (k+1)^2$ by the Induction Hypothesis
  $= k(k+1)(2k+1)/6 + (k+1)^2$ by definition of $SQ_c(k)$
  $= (k+1)(k+2)(2k+3)/6$ algebra
  $= SQ_c(k+1)$ by definition of $SQ_c(k+1)$
- Conclusion: $SQ_r(n) = SQ_c(n)$ for all $n \varepsilon 0$

# Another Example



- Prove that $0+1+...+n = n(n+1)/2$

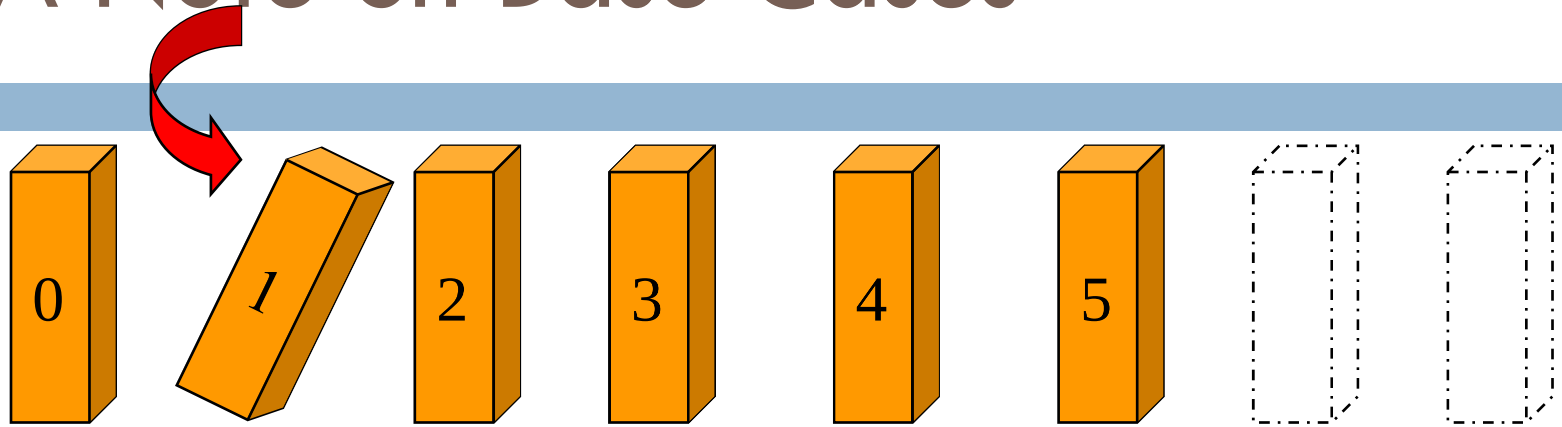
- Basis: Obviously holds for $n = 0$
- Induction Hypothesis: Assume $0+1+\ldots+k = k(k+1)/2$
- Inductive Step:

$$
\begin{aligned}
0+1+\ldots+(k+1) &= [0+1+\ldots+k] + (k+1) && \text{by def} \\
&= k(k+1)/2 + (k+1) && \text{by I.H.} \\
&= (k+1)(k+2)/2 && \text{algebra}
\end{aligned}
$$

- Conclusion: $0+1+\ldots+n = n(n+1)/2$ for all $n \geq 0$

# A Note on Base Cases






- Sometimes we are interested in showing some proposition is true for integers ≥ b
- Intuition: we knock over domino b, and dominoes in front get knocked over; not interested in 0,1,…,(b − 1)
- In general, the base case in induction does not have to be 0
- If base case is some integer b
  - Induction proves the proposition for n = b, b+1, b+2, …
  - Does not say anything about n = 0,1,…,b − 1

# Weak Induction: Nonzero Base Case



- Claim: You can make any amount of postage above 8¢ with some combination of 3¢ and 5¢ stamps
- Basis: True for 8¢: 8 = 3 + 5
- Induction Hypothesis: Suppose true for some $k \geq 8$
- Inductive Step:
  - If used a 5¢ stamp to make k, replace it by two 3¢ stamps. Get k+1.
  - If did not use a 5¢ stamp to make k, must have used at least three 3¢ stamps. Replace three 3¢ stamps by two 5¢ stamps. Get k+1.
- Conclusion: Any amount of postage above 8¢ can be made with some combination of 3¢ and 5¢ stamps

# What are the “Dominos”?

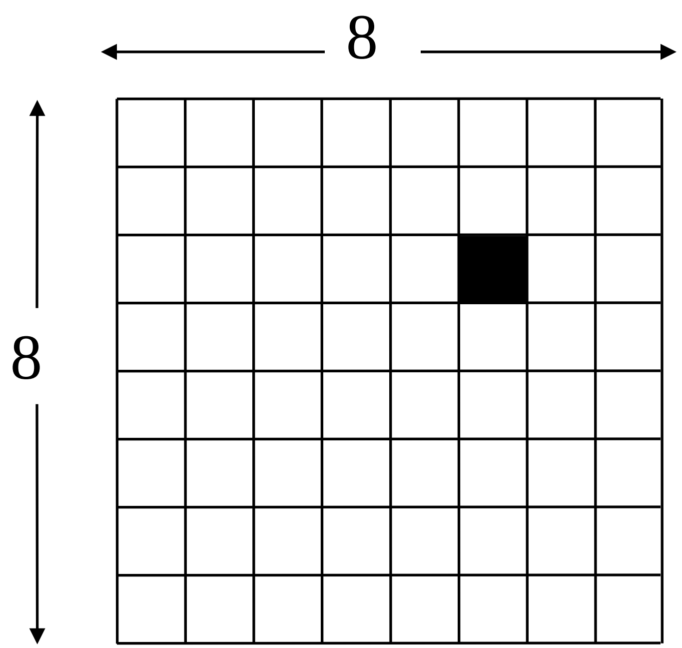


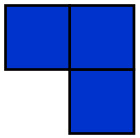
- In some problems, it can be tricky to determine how to set up the induction
- This is particularly true for geometric problems that can be attacked using induction

# A Tiling Problem



- A chessboard has one square cut out of it
- Can the remaining board be tiled using tiles of the shape shown in the picture (rotation allowed)?
- Not obvious that we can use induction!



# Proof Outline



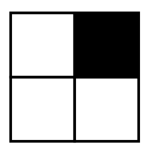
- Consider boards of size $2^n \times 2^n$ for n = 1,2,…
- Basis: Show that tiling is possible for 2 x 2 board
- Induction Hypothesis: Assume the $2^k \times 2^k$ board can be tiled
- Inductive Step: Using I.H. show that the $2^{k+1} \times 2^{k+1}$ board can be tiled
- Conclusion: Any $2^n \times 2^n$ board can be tiled, n = 1,2,…

  - Our chessboard (8 x 8) is a special case of this argument
  - We will have proven the 8 x 8 special case by solving a more general problem!

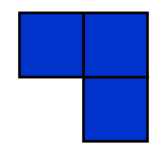
# Basis



- The 2 x 2 board can be tiled regardless of which one of the four pieces has been omitted



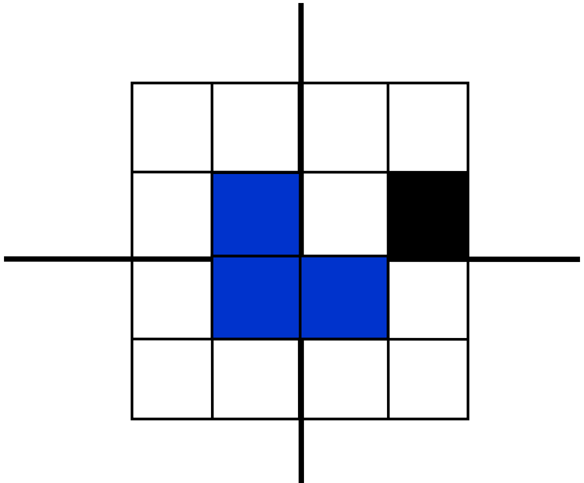
2 x 2 board

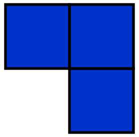


# 4 x 4 Case



- Divide the 4 x 4 board into four 2 x 2 sub-boards
- One of the four sub-boards has the missing piece
  - By the I.H., that sub-board can be tiled since it is a 2 x 2 board with a missing piece
- Tile center squares of three remaining sub-boards as shown
  - This leaves three 2 x 2 boards, each with a missing piece
  - We know these can be tiled by the Induction Hypothesis

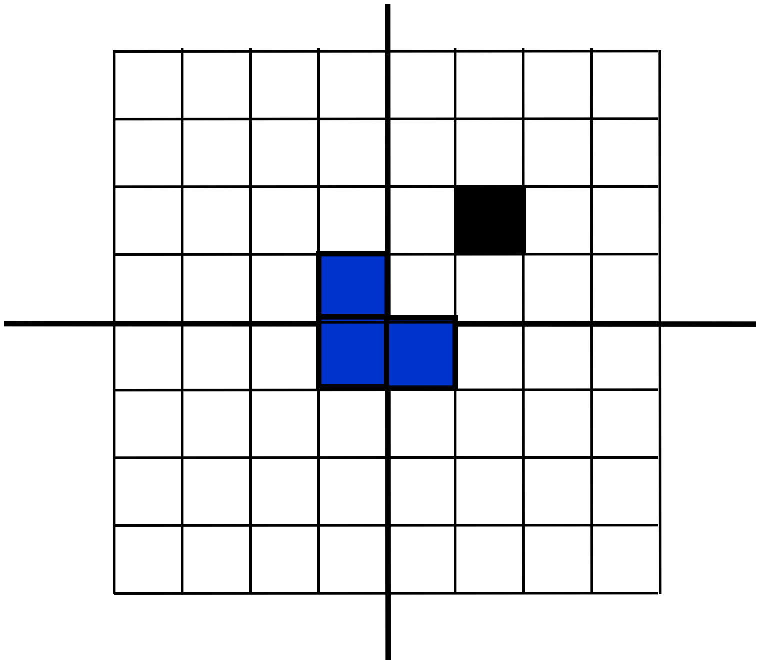


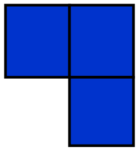


# $2^{k+1}$ x $2^{k+1}$ case



- Divide board into four sub-boards and tile the center squares of the three complete sub-boards
- The remaining portions of the sub-boards can be tiled by the I.H. (which assumes we can tile $2^k$ x $2^k$ boards)





# When Induction Fails



- Sometimes an inductive proof strategy for some proposition may fail
- This does not necessarily mean that the proposition is wrong
  - It may just mean that the particular inductive strategy you are using is the wrong choice
- A different induction hypothesis (or a different proof strategy altogether) may succeed

# Tiling Example (Poor Strategy)



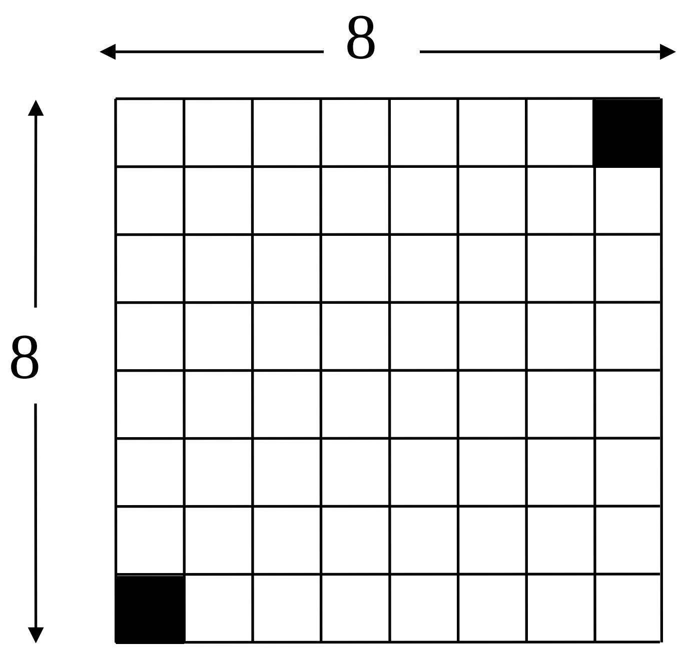
- Let's try a different induction strategy
- Proposition
  - Any n x n board with one missing square can be tiled
- Problem
  - A 3 x 3 board with one missing square has 8 remaining squares, but our tile has 3 squares; tiling is impossible
- Thus, any attempt to give an inductive proof of this proposition *must fail*

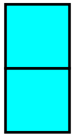
- Note that this failed proof does not tell us anything about the 8x8 case

# A Seemingly Similar Tiling Problem



- A chessboard has opposite corners cut out of it. Can the remaining board be tiled using tiles of the shape shown in the picture (rotation allowed)?
- Induction fails here. Why? (Well…for one thing, this board can't be tiled with dominos.)



# Strong Induction



- We want to prove that some property P holds for all n
- Weak induction
  - P(0): Show that property P is true for 0
  - P(k) ⇒ P(k+1): Show that if property P is true for k, it is true for k+1
  - Conclude that P(n) holds for all n
- Strong induction
  - P(0): Show that property P is true for 0
  - P(0) and P(1) and … and P(k) ⇒ P(k+1): show that if P is true for numbers less than or equal to k, it is true for k+1
  - Conclude that P(n) holds for all n
- Both proof techniques are equally powerful

# Conclusion



- Induction is a powerful proof technique
- Recursion is a powerful programming technique
- Induction and recursion are closely related
  - We can use induction to prove correctness and complexity results about recursive programs