Priority Queues and Heaps

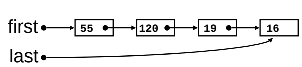
The Bag Interface

```
interface Bag<E> {
   void put(E obj);
   E get(); //extract some element
   boolean isEmpty();
}
```

Examples: Stack, Queue

Stacks and Queues as Lists

- Stack (LIFO) implemented as list
  - `put()`, `get()` from front of list
- Queue (FIFO) implemented as list
  - `put()` on back of list, `get()` from front of list
- All `Bag` operations are O(1)



Priority Queue

- A `Bag` in which data items are `Comparable`
- *lesser* elements (as determined by `compareTo()`) have *higher* priority
- `get()` returns the element with the highest priority = least in the `compareTo()` ordering
- break ties arbitrarily

Examples

- Scheduling jobs to run on a computer
  - default priority = arrival time
  - priority can be changed by operator
- Scheduling events to be processed by an event handler
  - priority = time of occurrence
- Airline check-in
  - first class, business class, coach
  - FIFO within each class

Priority Queues

```
interface Bag<E> {
   void put(E obj);
   E get(); //extract some element
   boolean isEmpty();
}

interface PriorityQueue<E extends Comparable>
   extends Bag<E> {}
```

Priority Queues as Lists

- Maintain as unordered list
  - `put()` puts new element at front – O(1)
  - `get()` must search the list – O(n)
- Maintain as ordered list
  - `put()` must search the list – O(n)
  - `get()` gets element at front – O(1)
- In either case, O(n^2) to process n elements

Can we do better?

Important Special Case

- Fixed number of priority levels 0,...,p – 1
- FIFO within each level
- Example: airline check-in

- `put()`– insert in appropriate queue – O(1)
- `get()`– must find a nonempty queue – O(p)

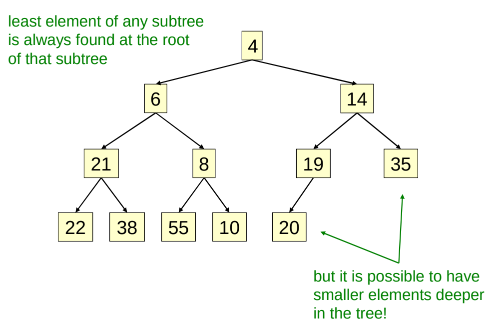
Heaps

- A *heap* is a concrete data structure that can be used to implement priority queues
- Gives better complexity than either ordered or unordered list implementation:
  - `put()`, `get()` – O(log n)
  - `isEmpty()` – O(1)
- O(n log n) to process n elements
- Do not confuse with *heap memory*, where the Java virtual machine allocates space for objects – different usage!

Heaps

- Binary tree with data at each node
- Satisfies the *Heap Order Invariant*:

The least (highest priority) element of any subtree is found at the root of that subtree



Examples of Heaps

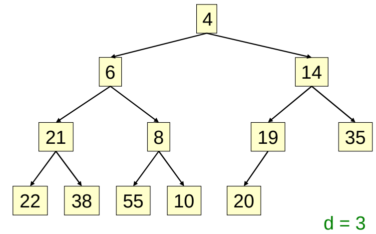
- Ages of people in family tree
  - parent is always older than children, but you can have an uncle who is younger than you
- Salaries of employees of a company
  - bosses generally make more than subordinates, but a VP in one subdivision may make less than a Project Supervisor in a different subdivision

Balanced Heaps

Two restrictions:

1. Any node of depth < d – 1 has exactly 2 children, where d is the height of the tree
   – implies that any two maximal paths (path from a root to a leaf) are of length d or d – 1, and the tree has at least 2^d nodes
2. All maximal paths of length d are to the left of those of length d – 1

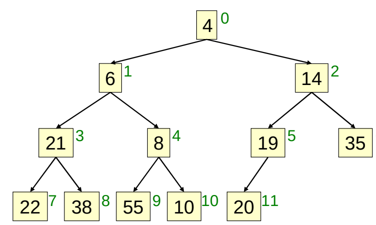
A Balanced Heap



Store in an Array or Vector

- Elements of the heap are stored in the array in order, going across each level from left to right, top to bottom
- The children of the node at array index n are found at 2n + 1 and 2n + 2
- The parent of node n is found at (n – 1)/2

Store in an Array or Vector

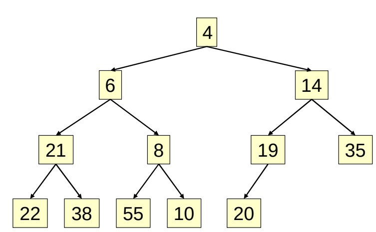


children of node n are found at 2n + 1 and 2n + 2

`put()`

- Put the new element at the end of the array
- If this violates heap order because it is smaller than its parent, swap it with its parent
- Continue swapping it up until it finds its rightful place
- The heap invariant is maintained!

`put()`



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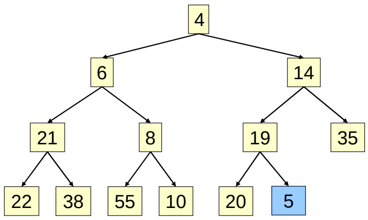

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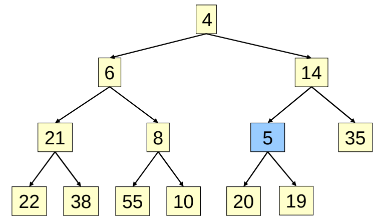

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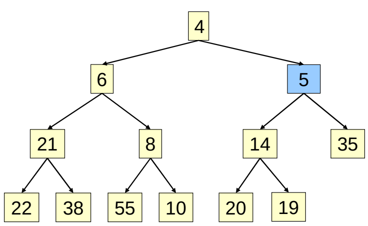

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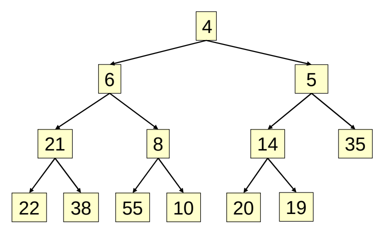

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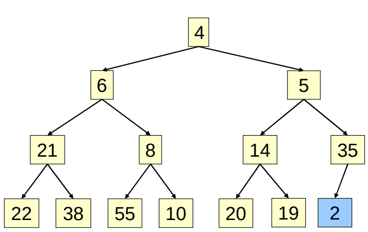

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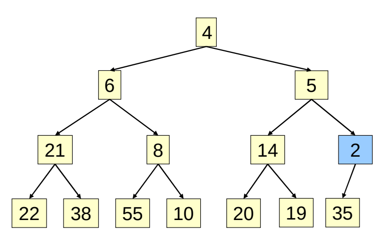

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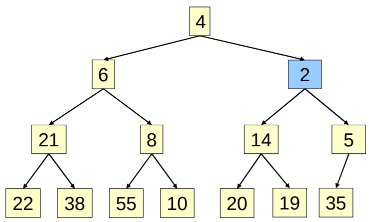

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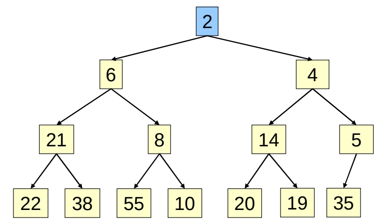

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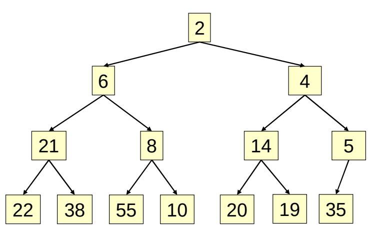


put()

- Time is O(log n), since the tree is balanced
  - size of tree is exponential as a function of depth
  - depth of tree is logarithmic as a function of size

put()

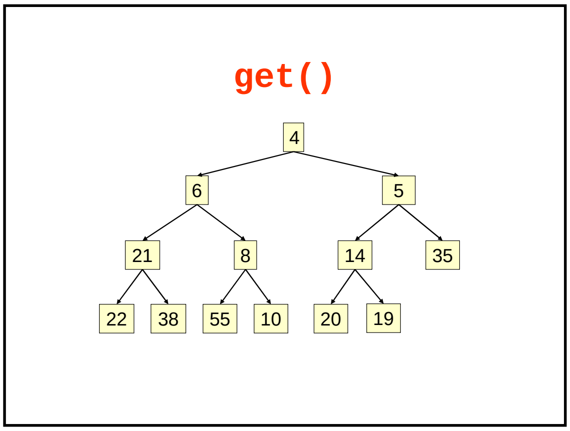
```
class PQ<E extends Comparable> extends java.util.Vector<E>
  implements PriorityQueue<E> {

  public void put(E obj) {
    add(obj); //add new element to end of array
    rotateUp(size() - 1);
  }

  private void rotateUp(int index) {
    if (index == 0) return;
    int parent = (index - 1)/2;
    if (elementAt(parent).compareTo(elementAt(index)) <= 0)
      return;
    swap(index, parent);
    rotateUp(parent);
  }
```

get()

- Remove the least element – it is at the root
- This leaves a hole at the root – fill it in with the last element of the array
- If this violates heap order because the root element is too big, swap it down with the smaller of its children
- Continue swapping it down until it finds its rightful place
- The heap invariant is maintained!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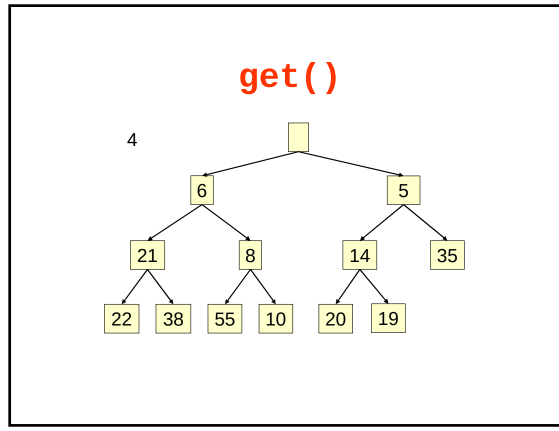
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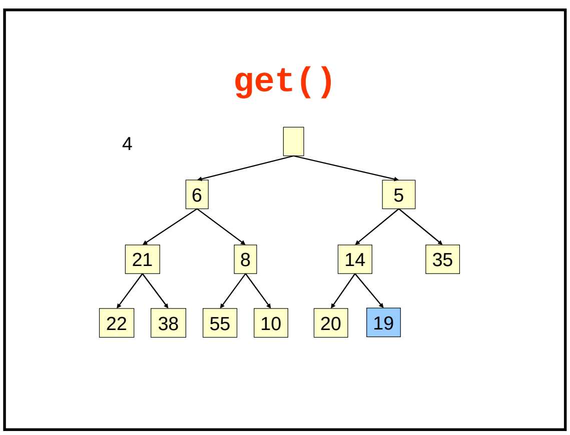
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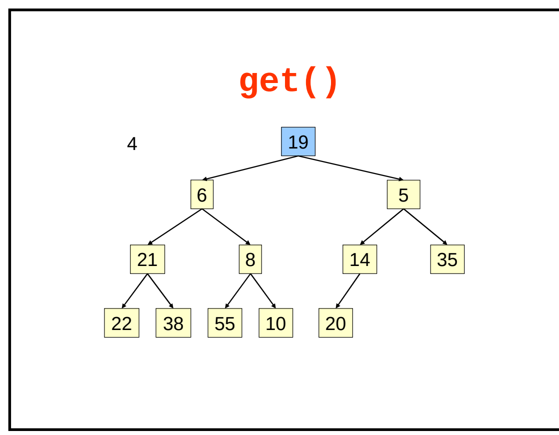
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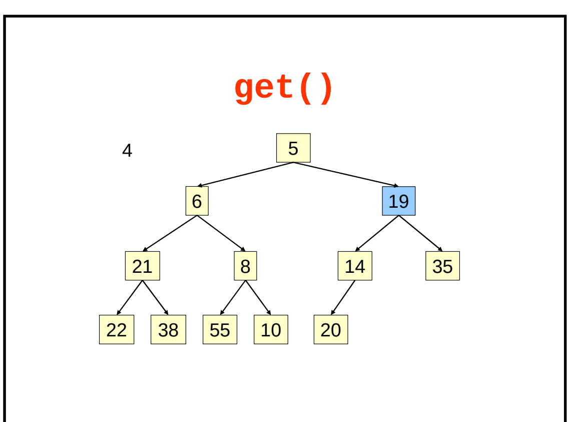
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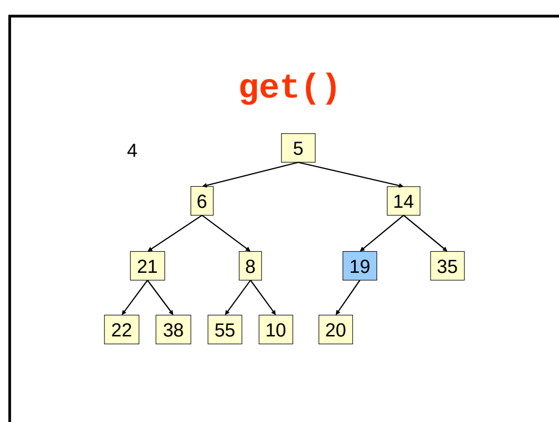
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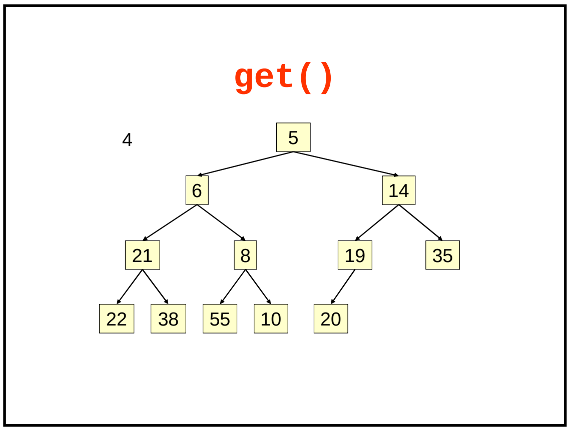
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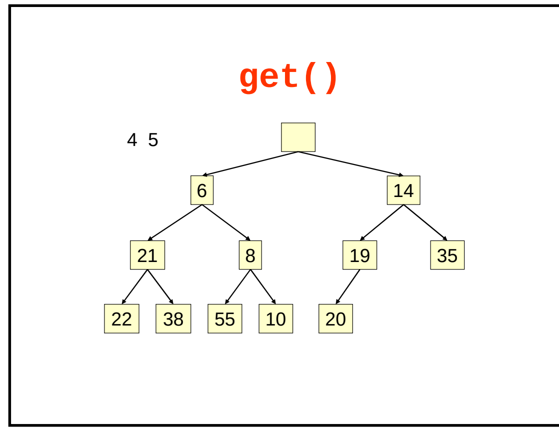
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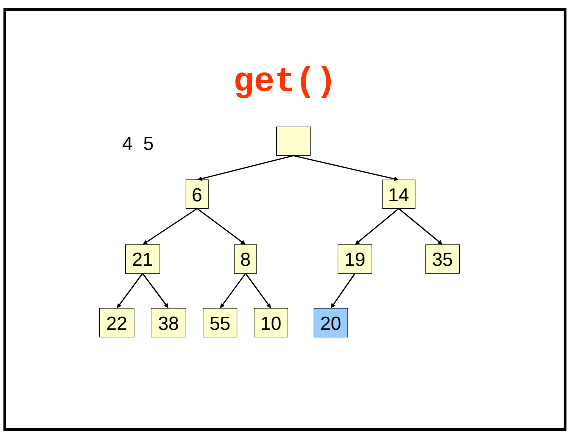
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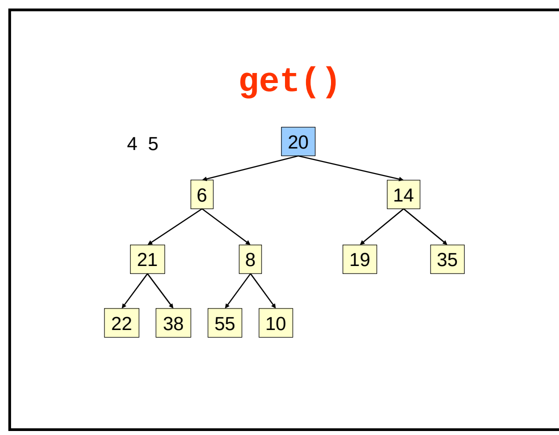
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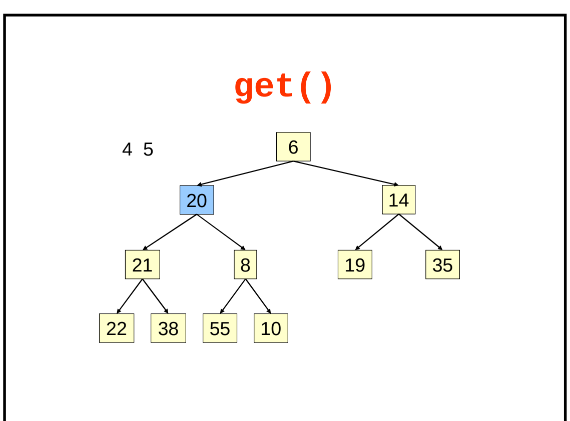
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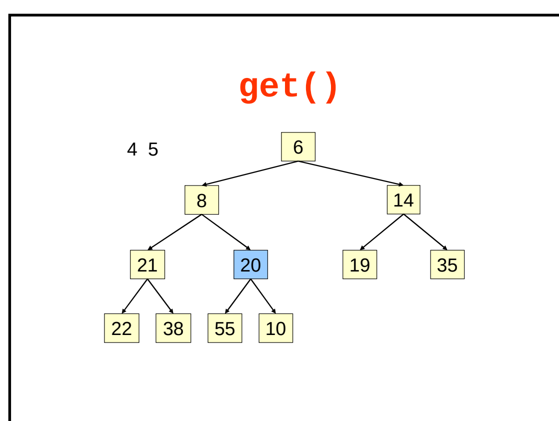
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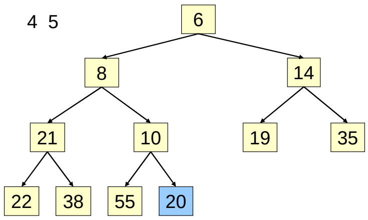

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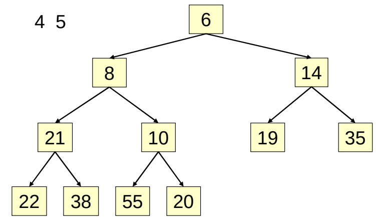

get()

- Time is O(log n), since the tree is balanced

get()

```
public E get() {
  if (isEmpty()) throw new NoSuchElementException();
  E temp = elementAt(0);
  setElementAt(elementAt(size() - 1), 0);
  setSize(size() - 1);
  rotateDown(0);
  return temp;
}
private void rotateDown(int index) {
  int child = 2*(index + 1); //right child
  if (child >= size()
    || elementAt(child - 1).compareTo(elementAt(child)) < 0)
      child -= 1;
  if (child >= size()) return;
  if (elementAt(index).compareTo(elementAt(child)) <= 0)
    return;
  swap(index, child);
  rotateDown(child);
}
```

HeapSort

Given a **`Comparable[]`** array of length n,

1. Put all n elements into a heap – O(n log n)
2. Repeatedly get the min – O(n log n)

```
public static void heapSort(Comparable[] a) {
   PriorityQueue<Comparable> pq = new PQ<Comparable>();
   for (Comparable x : a) { pq.put(x); }
   for (int i = 0; i < a.length; i++) { a[i] = pq.get(); }
}
```