

Induction

(by Nicholas Ruozzi)

Mathematical induction is a technique of proof that occurs quite frequently in computer science and mathematics. It is often used to demonstrate the correctness of recursive algorithms, but here the focus will just be on applying the method of induction to basic mathematical proofs. There are two types of induction, weak induction and strong induction. The difference between these two types is in the induction hypothesis. In weak induction, we assume only that if one case is true then the next case is also true. In strong induction, we assume that all cases before a particular case is true in order to show that the next case is true. These differences are best illustrated with examples.

Problem 1 (Weak Induction):

Let $S(n)$ = the sum of the first n odd numbers greater than 0.

Show that $S(n) = 1 + 3 + \dots + (2n - 1) = n^2$.

Base Case ($n = 1$): $S(1) = 1 = 1^2$. So the result holds for $n = 1$.

Induction Hypothesis: Assume that $S(k) = k^2$.

Show that $S(k + 1) = (k + 1)^2$.

$$\begin{aligned} S(k + 1) &= 1 + 3 + \dots + (2k - 1) + (2(k + 1) - 1) \\ &= S(k) + 2(k + 1) - 1 && \text{(by definition of } S(n)) \\ &= k^2 + 2(k + 1) - 1 && \text{(by the induction hypothesis)} \\ &= k^2 + 2k + 1 \\ &= (k + 1)(k + 1) && \text{(factoring)} \\ &= (k + 1)^2 \end{aligned}$$

Therefore, we can conclude that since $S(1) = 1$ and that $S(k) \Rightarrow S(k + 1)$ then $S(n)$ is equal to the sum of the first n odd numbers for all $n > 0$.

Problem 2 (Weak Induction):

Let $P(n) = n^3 - n$ for all $n \geq 0$.

Show that $P(n) = n^3 - n$ is divisible by 3.

Base Case ($n = 0$): $P(0) = 0^3 - 0 = 0$ which is divisible by 3.

Induction Hypothesis: Assume that $P(k)$ is divisible by 3.

Show $P(k + 1)$ is divisible by 3.

$$\begin{aligned} P(k + 1) &= (k + 1)^3 - (k + 1) \\ &= k^3 + 3k^2 + 3k + 1 - (k + 1) \\ &= k^3 - k + 3k^2 + 3k \\ &= P(k) + 3k^2 + 3k && \text{(by definition of } P(n)) \end{aligned}$$

$3(k^2 + k)$ is divisible by 3.
 $P(k)$ is divisible by 3. (by induction hypothesis)
 $\Rightarrow P(k) + 3k^2 + 3k$ is also divisible by 3.

Therefore, we can conclude that since $P(0)$ is divisible by 3 and that $P(k)$ is divisible by 3
 $\Rightarrow P(k + 1)$ is divisible by 3 then $P(n)$ is divisible by 3 for all $n \geq 0$.

Problem 3 (Faulty Induction):

Let H_n = Set of n horses.

Show that H_n contains all horses of the same color.

Base Case ($n = 1$): There is one horse, so all of the horses have the same color.

Induction Hypothesis: Assume that H_k contains k horses of the same color.

Show that H_{k+1} contains horses of the same color.

Let S_1 be a subset of H_{k+1} that contains k horses by removing one horse from H_{k+1} .

Let S_2 be a subset of H_{k+1} that contains k horses by removing a different horse from H_{k+1} than before.

S_1 has k horses of the same color by the inductive hypothesis.

S_2 has k horses of the same color by the inductive hypothesis.

Now, since all of S_1 is the same color, and the sets have at least one horse in their intersection (some of the horses in S_1 are in S_2), and S_2 contains horses of all the same color, then S_1 and S_2 contain horses of the same color.

Therefore, all horses of H_{k+1} are the same color!

This clearly cannot be correct. Intuition tells us that all horses do not have the same color. Think of the case H_2 : There are only two horses, so if we divide H_2 into two sets, each set will have one horse, but there are no horses in common, so the horses do not need to have the same color.

Problem 4 (Strong Induction):

Show that for all integers $n \geq 2$, n can be factored into prime numbers.

Base Case ($n = 2$): 2 is a prime number and as a result is its own prime factorization.

Induction Hypothesis: Assume that there exists a prime factorization for all n such that ($2 \leq n \leq k$) (*This is the part that makes this proof strong induction*). Show that $k + 1$ can be factored into prime numbers.

If $(k + 1)$ is a prime number then $(k + 1)$ is its prime factorization.

Otherwise, $(k + 1) = r * s$ where r and s are integers between 2 and k inclusive.

$r < (k + 1)$ implies that it has a prime factorization such that $r = \prod p_i$ where p_i are the prime factors of r .

$s < (k + 1)$ implies that it has a prime factorization such that $s = \prod q_j$ where q_j are the prime factors of s .

Therefore, $(k + 1) = (\prod p_i)(\prod q_j)$.

This implies that $(k + 1)$ can be factored into primes.

In conclusion, 2 can be factored into primes and if all numbers less than k can be factored into primes then it has been show that $(k + 1)$ can also be factored into primes.

Therefore, for all $n \geq 2$, n can be factored into prime numbers.

Problem 5 (Weak Induction with Multiple Base Cases):

Show that you can make any amount of stamps greater than 8-cents from any amount of 3-cent and 5-cent stamps.

Base Cases:

8-cents: $8 = 3 + 5$

9-cents: $9 = 3 + 3 + 3$

10-cents: $10 = 5 + 5$

Inductive hypothesis: Assume you can make $(k - 3)$ -cents from 3-cents and 5-cents. Show that you can make k -cent stamps.

$$\begin{aligned} k\text{-cents} &= k + 3 - 3 \\ &= (k - 3) + 3 \end{aligned}$$

A $k-3$ can be made from 3 and 5 cent stamps by inductive assumption. This implies that if a $(k-3)$ -cent stamp can be made of 5 and 3 cent stamps then so can a k -cent stamp.

In conclusion, since 8, 9, and 10 cent stamps can all be manufactured then so can all n -cent stamps since if $(k-3)$ -cent stamps can be created then so can k -cent stamps and the result holds for three sequential base cases.