Optimization and least squares

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Administrivia

- A5 Part 1 due tomorrow by 5pm (please sign up for a demo slot)
- Part 2 will be due in two weeks (4/17)
- Prelim 2 on Tuesday 4/7 (in class)
  - Covers everything since Prelim 1, including:
  - Polygons, convex hull, interpolation, image transformation, feature-based object recognition, solving for affine transformations
  - Review session on Monday, 7pm, Upson 315

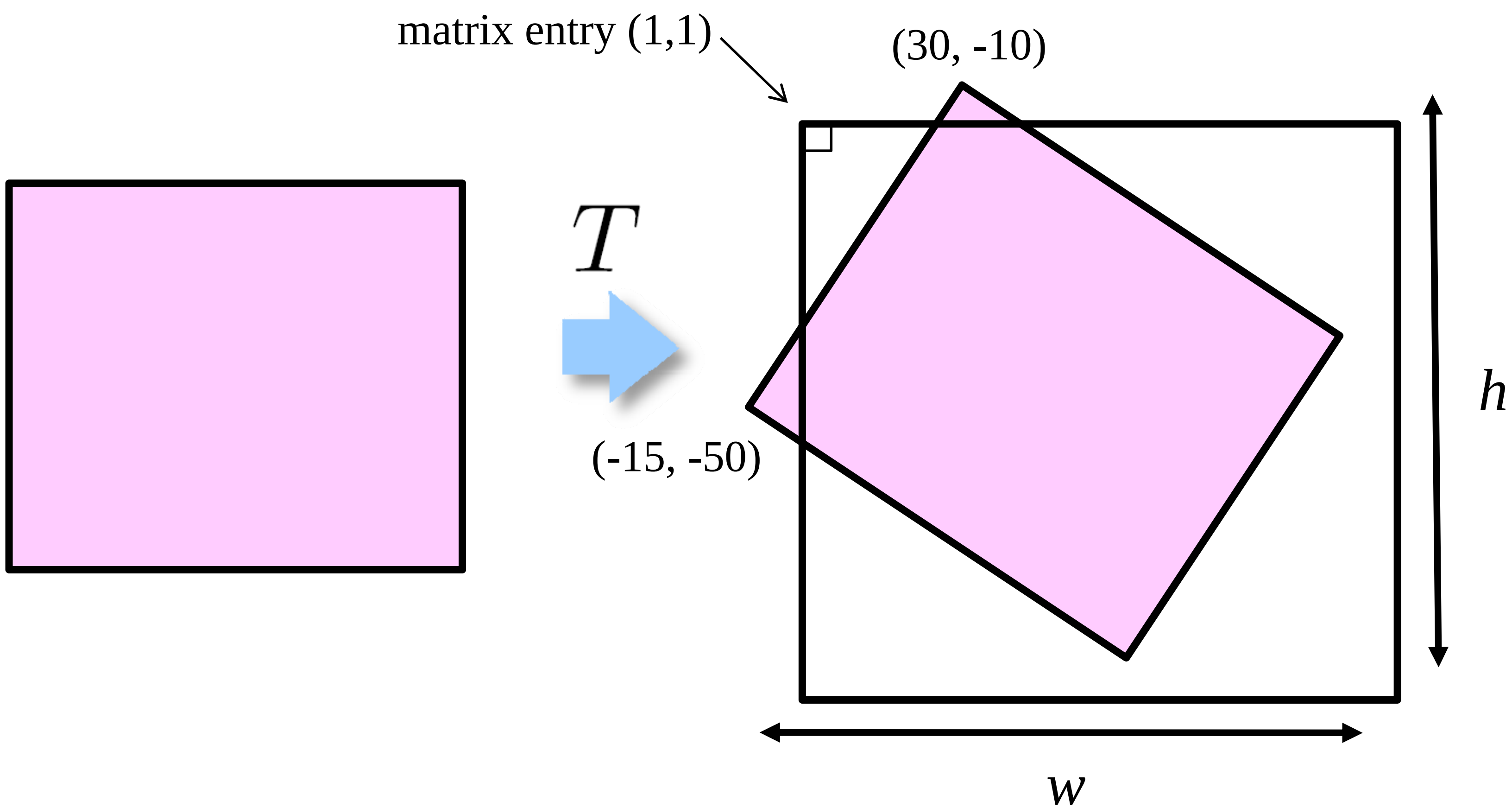
Image transformations

- What happens when the image moves outside the image boundary?
- Previously, we tried to fix this by changing the transformation: $T_2 R T_1$

Image transformations

- Another approach:
  - We can compute where the image moves to
    - in particular the minimum row and the minimum column
    - as well as the width and height of the output image (e.g., max_col – min_col + 1)
  - Then we can "shift" the image so it fits inside the output image
  - This could be done with another transformation, but could also just add/subtract min_col and min_row

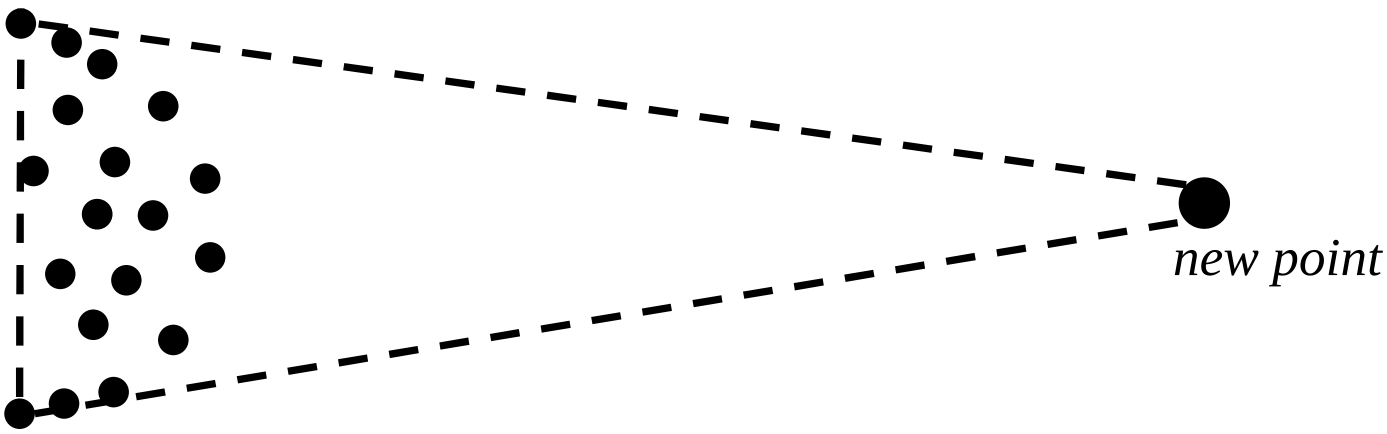
Shifting the image



- need to shift (row, col) by adding (min_row, min_col), before applying T^{-1}

Convex hulls

- If I have 100 points, and 10 are on the convex hull
- If I add 1 more point
  - the max # of points on the convex hull is 11
  - the min # of points on the convex hull is 3

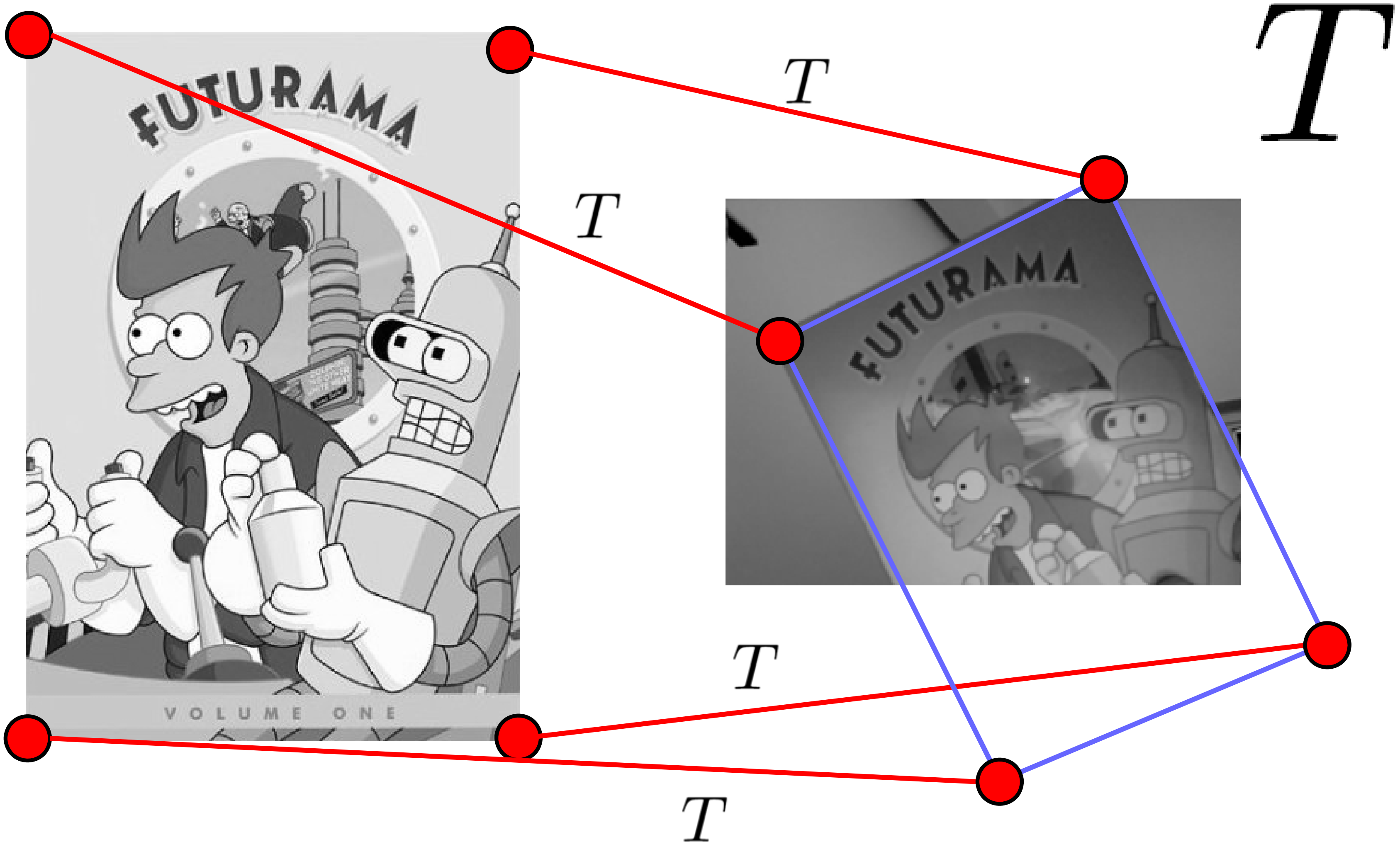


Object recognition

1. Detect features in two images
2. Match features between the two images

3. Select three matches at random
4. Solve for the affine transformation T
5. Count the number of inlier matches to T
6. If T is has the highest number of inliers so far, save it
7. Repeat 3-6 for N rounds, return the best T

Object recognition



When this won’t work

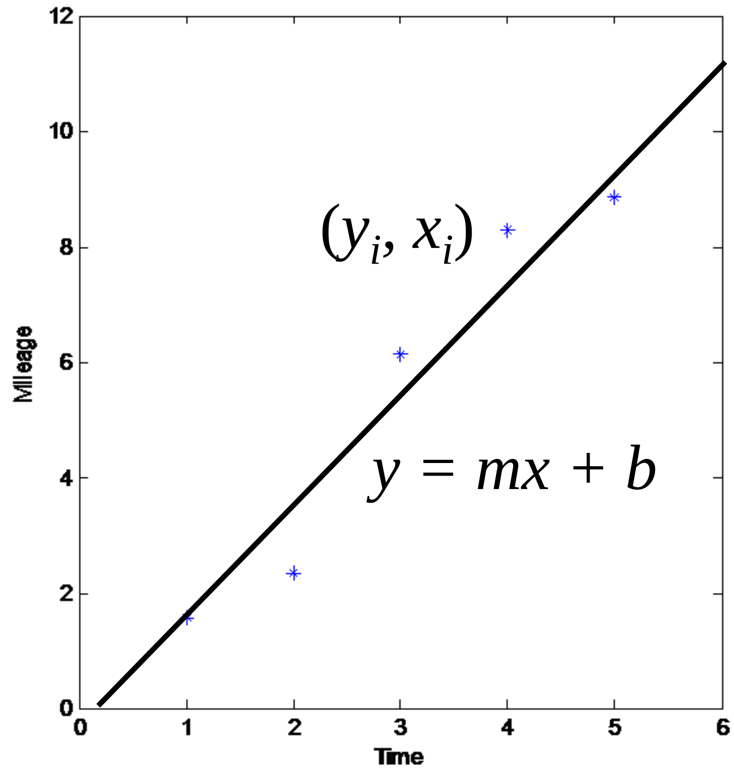
- If the percentage of inlier matches is small, then this may not work
- In theory, < 50% inliers could break it
  - But all the outliers would have to fit a single transform
- Often works with fewer inliers

A slightly trickier problem

- What if we want to fit *T* to more than three points?
  - For instance, all of the inliers we find?

- Say we found 100 inliers
- Now we have 200 equations, but still only 6 unknowns
- *Overdetermined* system of equations
- This brings us back to linear regression

Linear regression, > 2 points



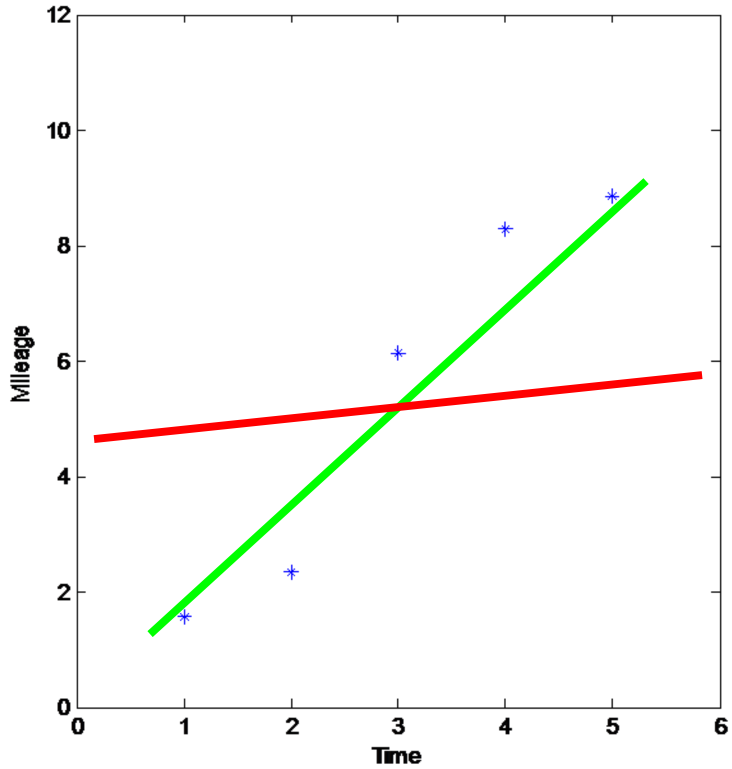
- The line won't necessarily pass through any data point

Some new definitions

- No line is perfect – we can only find the *best* line out of all the imperfect ones
- We'll define a function *Cost*(*m*,*b*) that measures how far a line is from the data, then find the best line
  - I.e., the (*m*,*b*) that minimizes Cost(*m*,*b*)
  - Such a function Cost(*m*,*b*) is called an *objective function*
  - You learned about these in section yesterday

Line goodness

- What makes a line good versus bad?
  - This is actually a very subtle question

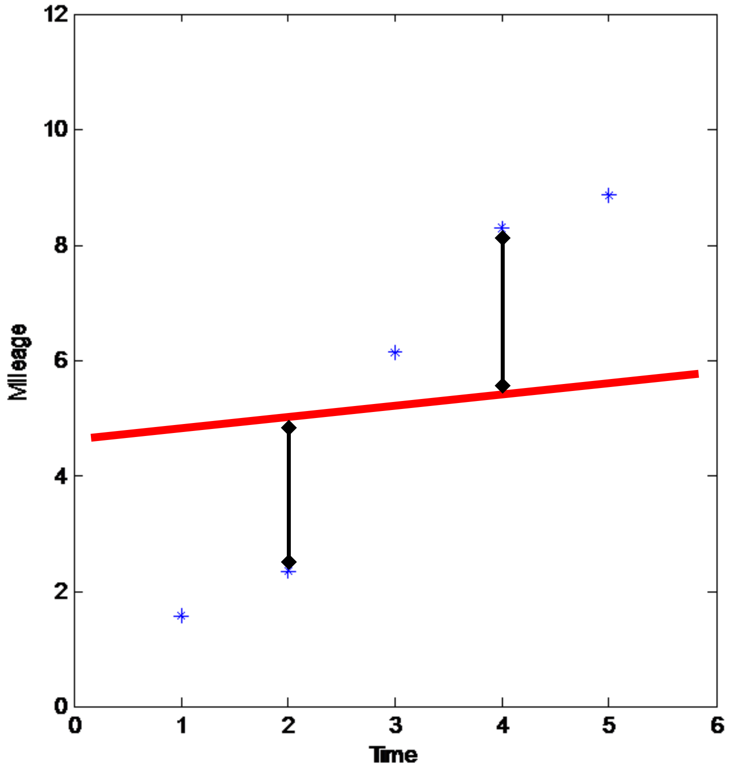


Residual errors

- The difference between what the model predicts and what we observe is called a residual error (i.e., a left-over)
  – Consider the data point (x,y)
  – The model m,b predicts (x,mx+b)
  – The residual is y – (mx + b)

- For 1D regressions, residuals can be easily visualized
  – Vertical distance to the line

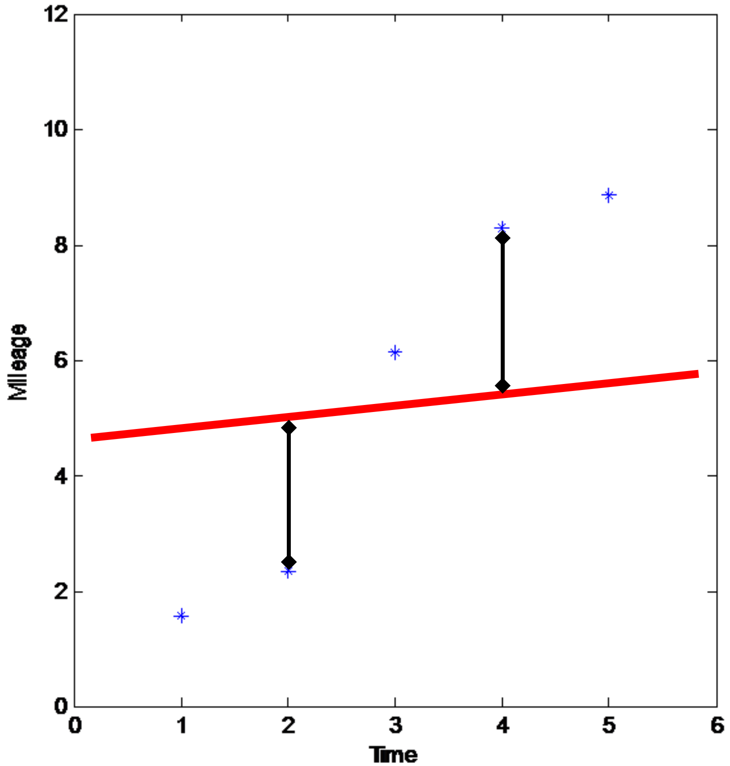
Least squares fitting



This is a reasonable cost function, but we usually use something slightly different

$$\text{Cost}(m,b) = \sum_{i=1}^{n} |y_i - (mx_i + b)|$$

Least squares fitting



We prefer to make this a **squared** distance

Called "least squares"

$$\text{Cost}(m, b) = \sum_{i=1}^{n} |y_i - (mx_i + b)|^2$$

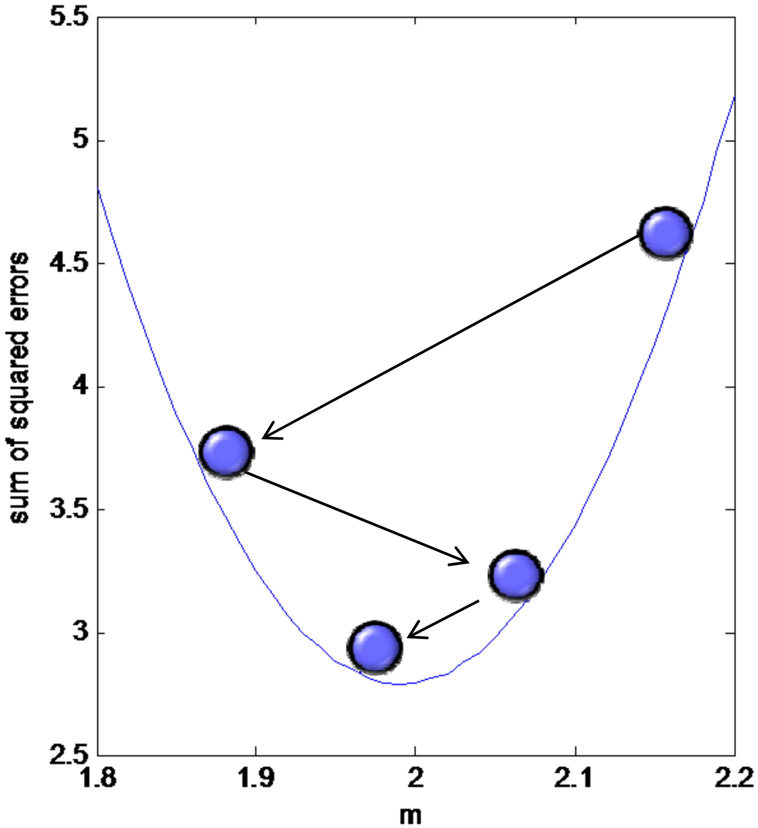
Why least squares?

- There are lots of reasonable objective functions
- Why do we want to use least squares?
- This is a very deep question
  - We will soon point out two things that are special about least squares
  - The full story probably needs to wait for graduate-level courses, or at least next semester

Gradient descent

- You learned about this in section
- Basic strategy:
  1. Start with some guess for the minimum
  2. Find the direction of steepest descent (gradient)
  3. Take a step in that direction (making sure that you get lower, if not, adjust the step size)
  4. Repeat until taking a step doesn’t get you much lower

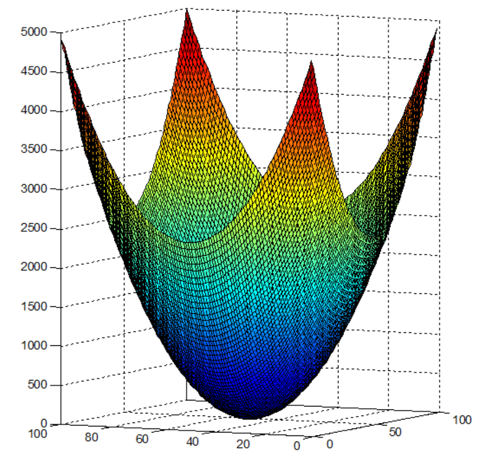
Gradient descent, 1D quadratic



- There is some black magic in setting the step size

Some error functions are easy

- A (positive) quadratic is a **convex** function
  - The set of points above the curve forms a (infinite) convex set
  - The previous slide shows this in 1D
    - But it's true in any dimension
- A sum of convex functions is convex
- Thus, the sum of squared error is convex
- Convex functions are "nice"
  - They have a single global minimum
  - Rolling downhill from anywhere gets you there



Consequences

- Our gradient descent method will always converge to the right answer
  - By slowly rolling downhill
  - It might take a long time, hard to predict exactly how long (see CS3220 and beyond)

What else about LS?

- Least squares has an even more amazing property than convexity
  - Consider the linear regression problem

$$\text{Cost}(m, b) = \sum_{i=1}^{n} |y_i - (mx_i + b)|^2$$

- There is a magic formula for the optimal choice of (m,b)
  - You don't need to roll downhill, you can "simply" compute the right answer

Closed-form solution!

- This is a huge part of why everyone uses least squares
- Other functions are convex, but have no closed-form solution, e.g.

$$\text{Cost}(m, b) = \sum_{i=1}^{n} |y_i - (mx_i + b)|$$

♦ Closed form LS formula

- The derivation requires linear algebra
  - Most books use calculus also, but it's not required (see the "Links" section on the course web page)

$$S_{xx} \equiv \sum_i (x_i - \bar{x})^2$$

$$S_{xy} \equiv \sum_i (x_i - \bar{x})(y_i - \bar{y})$$

$$m = \frac{S_{xy}}{S_{xx}}$$

$$b = \bar{y} - m\bar{x}$$

  - There's a closed form for *any* linear least-squares problem

Linear least squares

- Any formula where the residual is *linear* in the variables

- Examples:
  1. simple linear regression: $[y - (mx + b)]^2$
  2. finding an affine transformation

$$[x' - (ax + by + c)]^2 + [y' - (dx + ey + f)]^2$$

- Non-example:

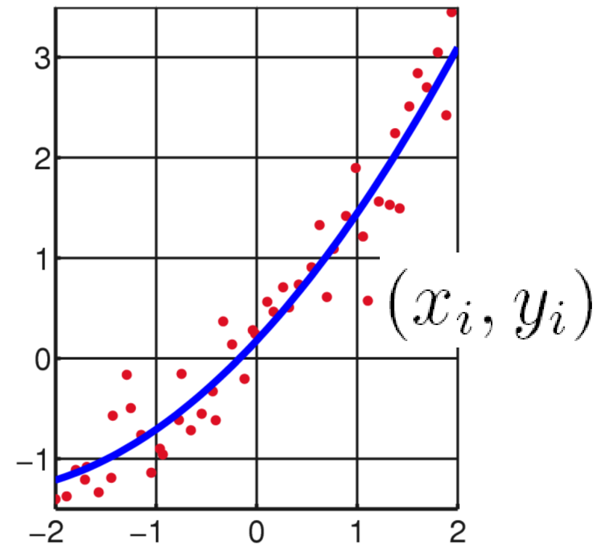
$$[x' - abc\ x]^2 \quad \text{(variables: a, b, c)}$$

Linear least squares

- Surprisingly, fitting the coefficients of a quadratic is still linear least squares

- The residual is still linear in the coefficients

  $\beta_1, \beta_2, \beta_3$



$$y = \beta_1 + \beta_2 x + \beta_3 x^2$$

$$\text{Cost}(\beta_1, \beta_2, \beta_3) = \sum_{i=1}^{n} |y_i - (\beta_1 + \beta_2 x + \beta_3 x^2)|^2$$

Wikipedia, "Least squares fitting"

Optimization

- Least squares is an example of an optimization problem
- Optimization: define a cost function and a set of possible solutions, find the one with the minimum cost
- Optimization is a *huge* field

Optimization strategies

- From worse to pretty good:

1. Guess an answer until you get it right
2. Guess an answer, improve it until you get it right
3. Magically compute the right answer

- For most problems 2 is the best we can do
- Some problems are easy enough that we can do 3

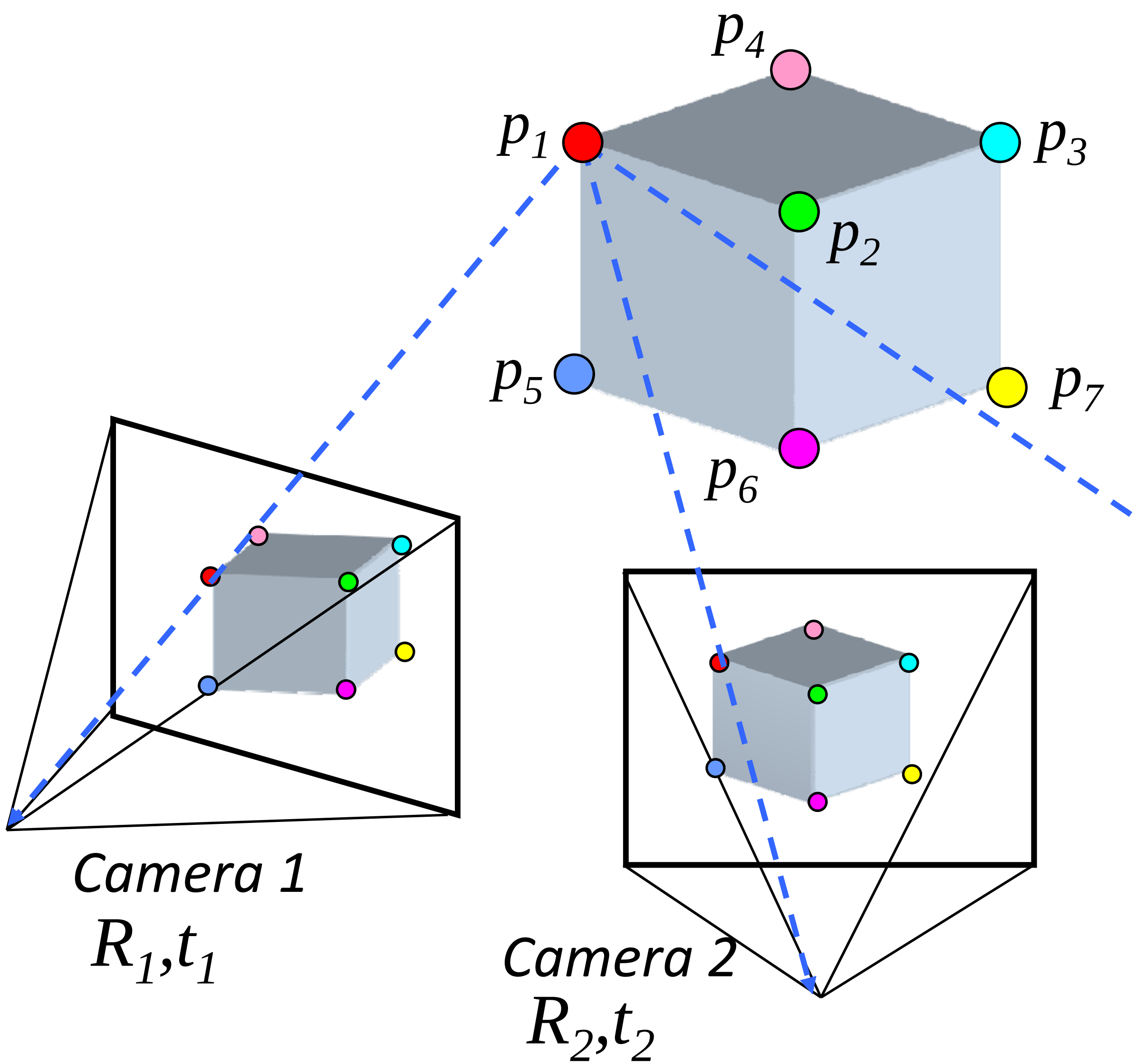
- We've seen several examples of this

Sorting as optimization

- Set of allowed answers: permutations of the input sequence
- Cost(permutation) = number of out-of-order pairs

- Algorithm 1: Snailsort
- Algorithm 2: Bubble sort
- Algorithm 3: ???

Another example: "structure from motion"



minimize

$$f\,(\mathbf{R}, \mathbf{T}, \mathbf{P})$$

Optimization is everywhere

- How can you give someone the smallest number of coins back as change?

- How do you schedule your classes so that there are the fewest conflicts?
- How do you manage your time to maximize the grade / work tradeoff?

Next time: Prelim 2