Lecture 16

More Recursion

Announcements for This Lecture

Prelim 1

- Prelim 1 back **today!**
  - Pick up in Gates 216
  - Solution posted in CMS
  - **Mean**: 78, **Median**: 83
- What are letter grades?
  - A bit too early to tell
  - **A**: Could be a consultant
  - **B**: Could take 2110
  - **C**: Good enough to pass

Assignments and Labs

- Need to be working on A4
  - Instructions are posted
  - Just reading it takes a while
  - Slightly longer than A3
  - Problems are harder
- **Lab Today**: lots of practice!
  - First 4 functions mandatory
  - Many optional ones in PDF
  - Exam questions on Prelim 2

Recall: Divide and Conquer

Goal: Solve problem P on a piece of data

data

Idea: Split data into two parts and solve problem

data 1	**data 2**
Solve Problem P	Solve Problem P

Combine Answer!

Example: Reversing a String

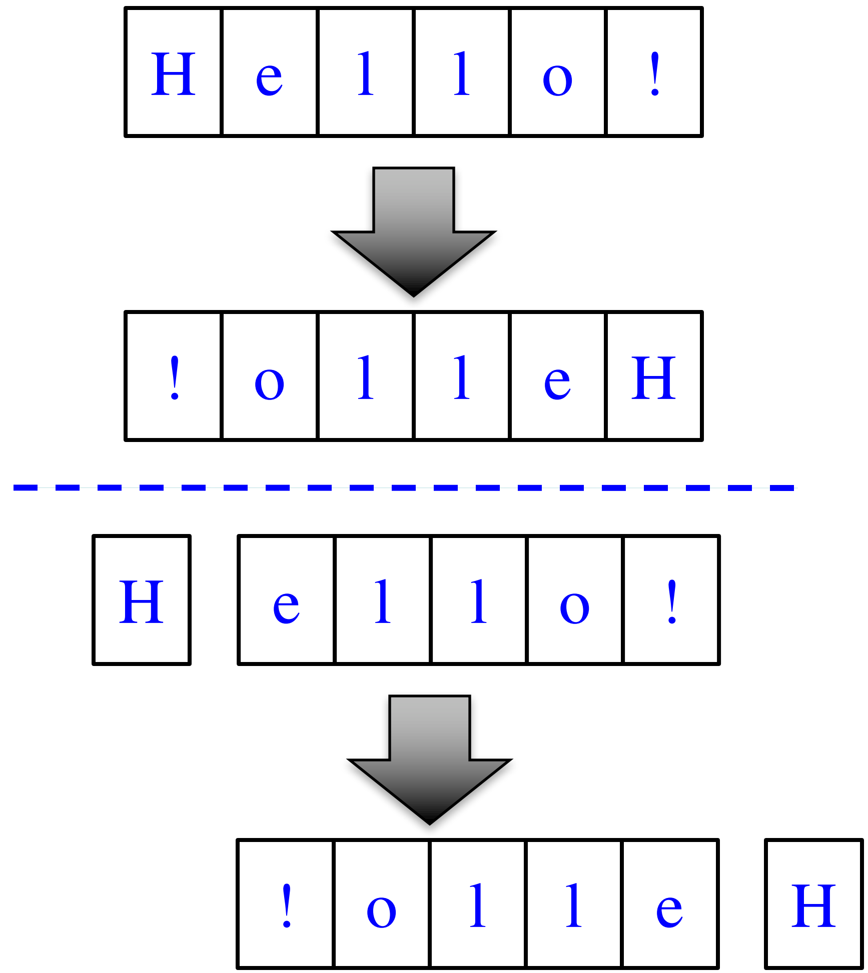
```
def reverse(s):
    """Returns: reverse of s

    Precondition: s a string"""
    # 1. Handle small data
    if len(s) <= 1:
        return s

    # 2. Break into two parts



    # 3. Combine the result
```



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    right = reverse(s[1:])

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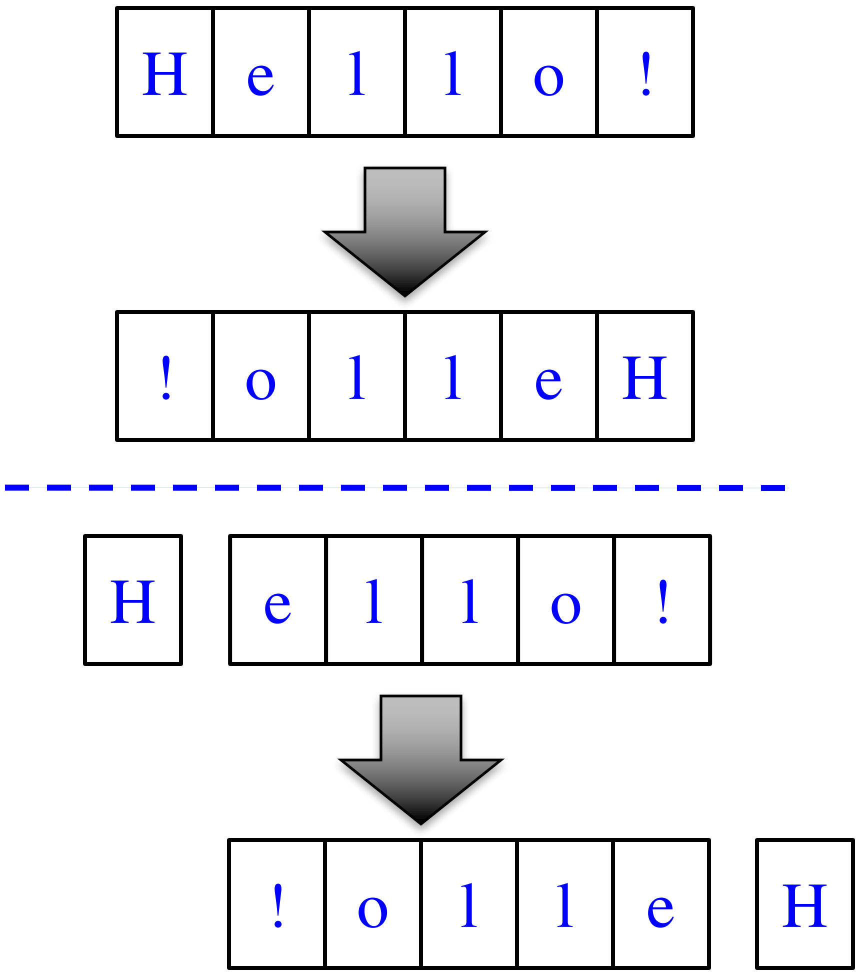

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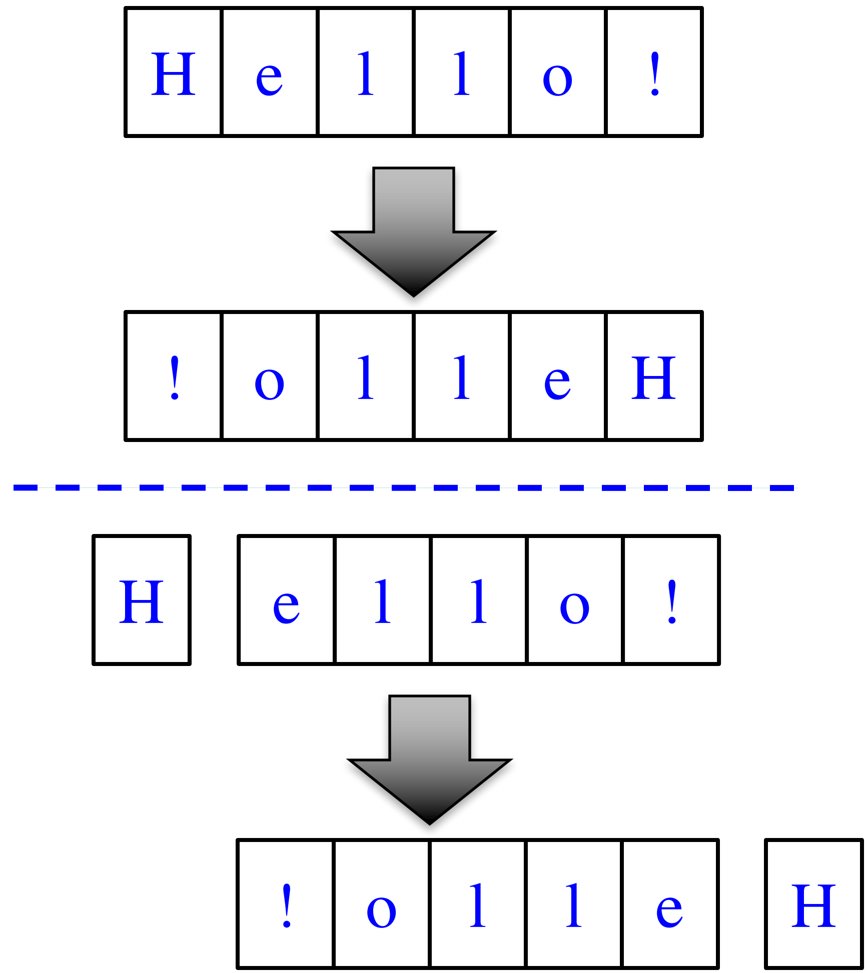
    # 3. Combine the result
    return right+left
```



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Base Case

Recursive Case

Example: Reversing a String

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    # 2. Break
    left   = s[0]
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    return right+left
```

Base Case

Remove recursive call

Recursive Case

How to Break Up a Recursive Function?

```
def commafy(s):
    """Returns: string with commas every 3 digits
    e.g. commafy('5341267') = '5,341,267'
    Precondition: s represents a non-negative int"""
```

Approach 1

5	341267

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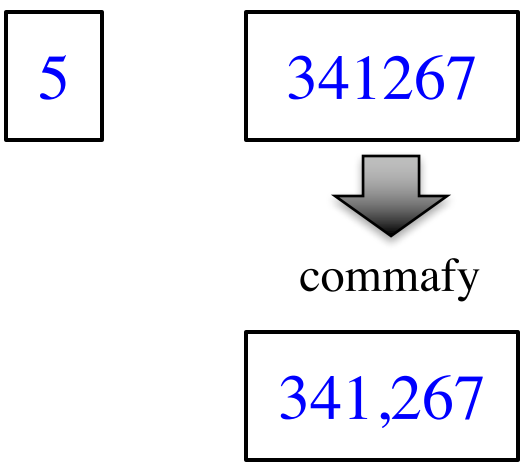

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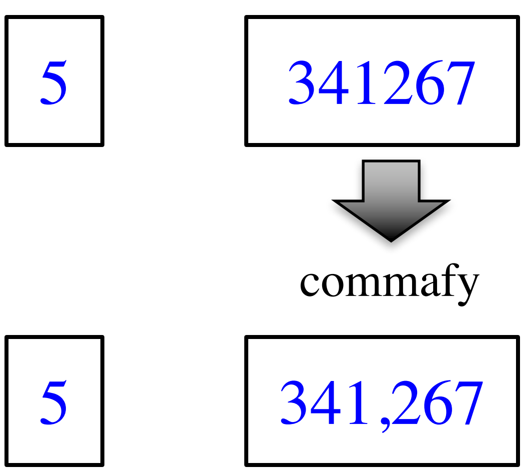

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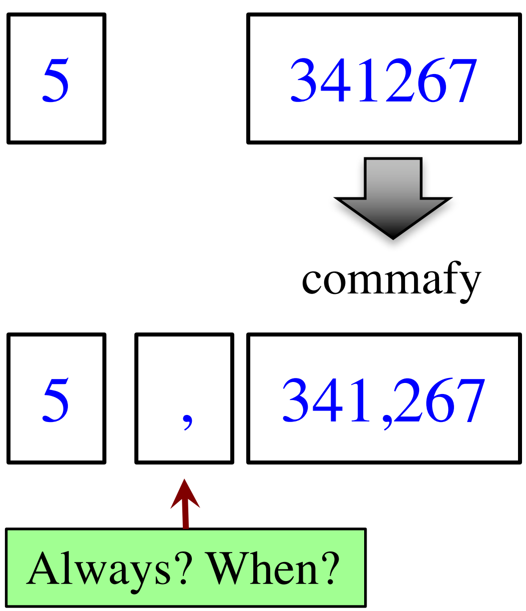

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```

Approach 1

5 | 341267

commafy

5 | , | 341,267

Always? When?

Approach 2

5341 | 267

How to Break Up a Recursive Function?

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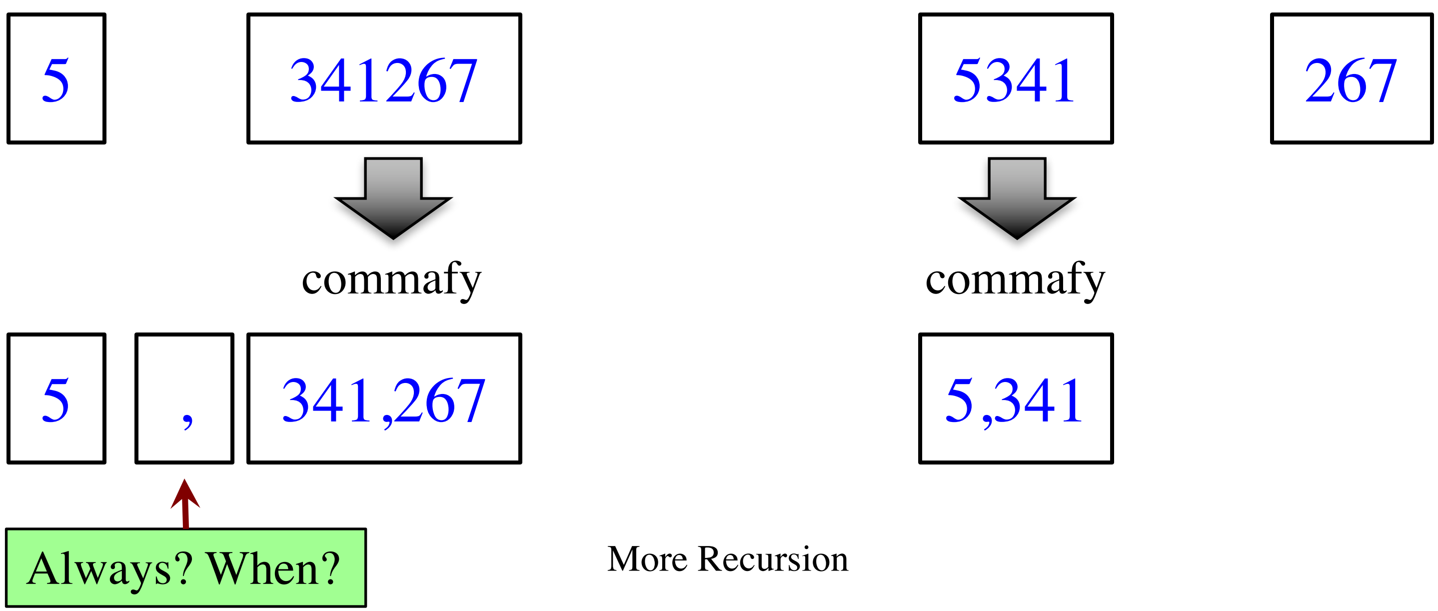
Approach 1

Approach 2



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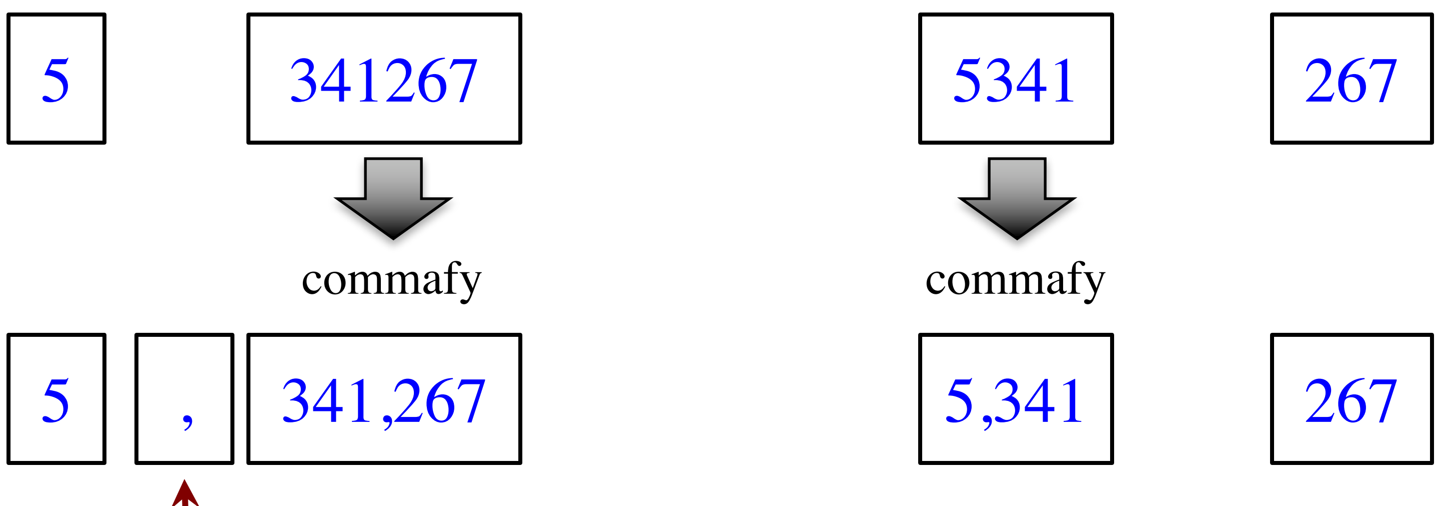
Approach 1

Approach 2



Always? When?

How to Break Up a Recursive Function?

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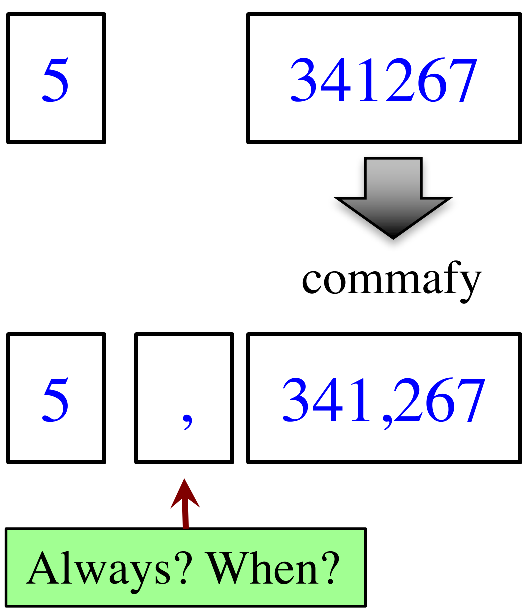
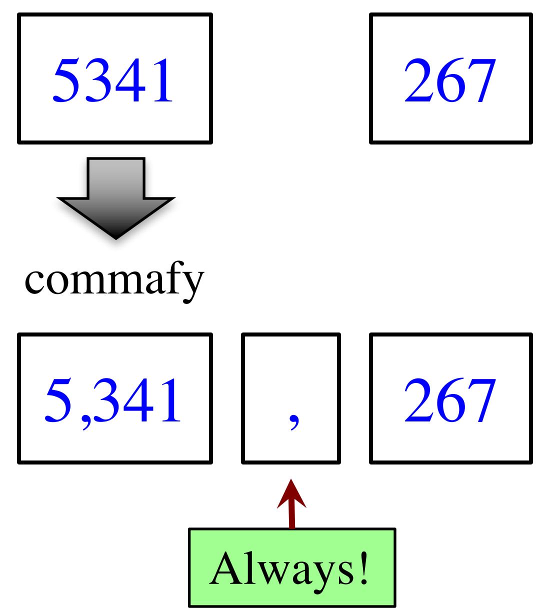


Approach 2



How to Break Up a Recursive Function?

```
def commafy(s):
    """Returns: string with commas every 3 digits
   e.g. commafy('5341267') = '5,341,267'
   Precondition: s represents a non-negative int"""
   # 1. Handle small data.
   if len(s) <= 3:
       return s

   # 2. Break into two parts
   left  = commafy(s[:-3])
   right = s[-3:]  # Small part on RIGHT

   # 3. Combine the result
   return left + ',' + right
```

Base Case

Recursive Case

How to Break Up a Recursive Function?

```
def exp(b, c)
    """Returns: b^c
    Precondition: b a float, c ≥ 0 an int"""
```

Approach 1

$$12^{256} = 12 \times (\boxed{12^{255}})$$

Recursive

$$b^c = b \times (b^{c-1})$$

Approach 2

$$12^{256} = (\boxed{12^{128}}) \times (\boxed{12^{128}})$$

Recursive

Recursive

$$b^c = (b \times b)^{c/2} \text{ if } c \text{ even}$$

Raising a Number to an Exponent

Approach 1

```
def exp(b, c)
    """Returns: b^c
    Precond: b a float, c ≥ 0 an int"""
    # b^0 is 1
    if c == 0:
        return 1

    # b^c = b(b^(c-1))
    left = b
    right = exp(b,c-1)

    return left*right
```

Approach 2

```
def exp(b, c)
    """Returns: b^c
    Precond: b a float, c ≥ 0 an int"""
    # b^0 is 1
    if c == 0:
        return 1

    # c > 0
    if c % 2 == 0:
        return exp(b*b,c/2)

    return b*exp(b*b,(c-1)/2)
```

Raising a Number to an Exponent

Approach 1

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    return left*right
```

Approach 2

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def exp(b, c)
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    # b^0 is 1
    if c == 0:
        return 1

    # c > 0
    if c % 2 == 0:
        return exp(b*b,c/2)

    return b*exp(b*b,(c-1)/2)
```

left right

left right

Raising a Number to an Exponent

```
def exp(b, c)
    """Returns: b^c
    Precond: b a float, c ≥ 0 an int"""
    # b^0 is 1
    if c == 0:
        return 1

    # c > 0
    if c % 2 == 0:
        return exp(b*b,c/2)
    return b*exp(b*b,(c-1)/2)
```

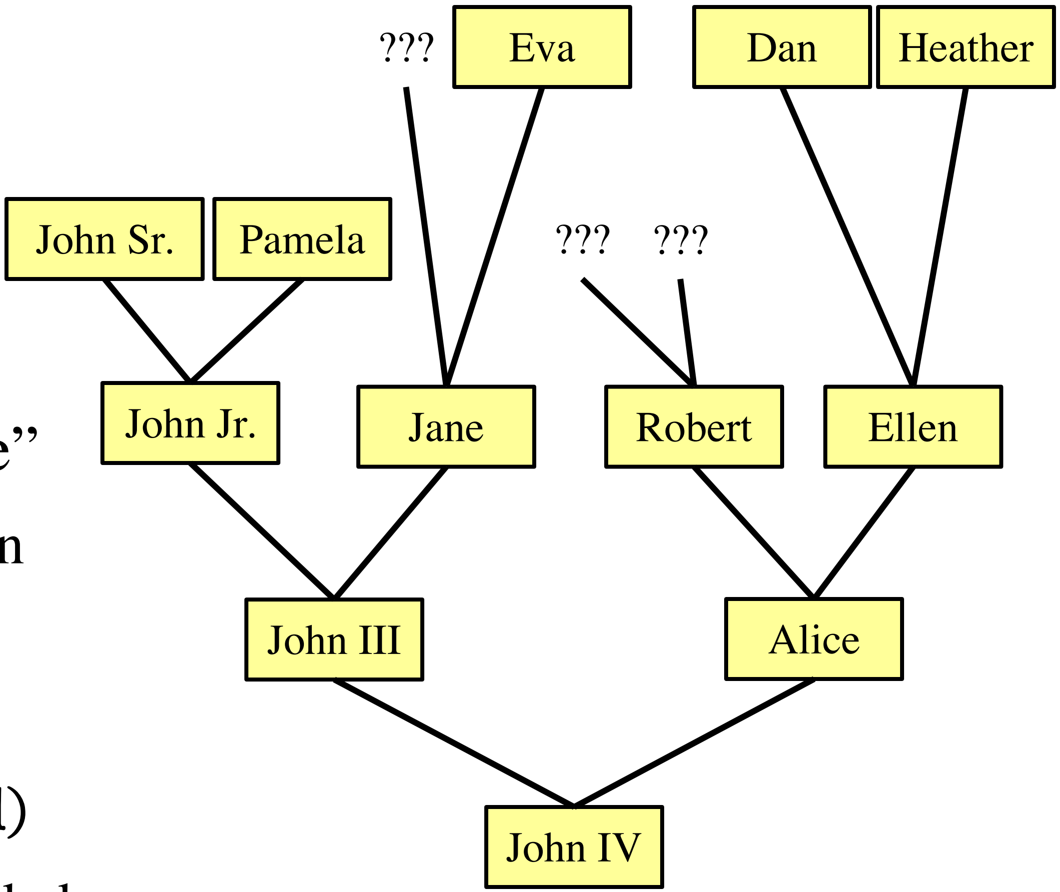
c	# of calls
0	0
1	1
2	2
4	3
8	4
16	5
32	6
2^n	n + 1

32768 is 215

b^{32768} needs only 215 calls!

Recursion and Objects

- Class Person (`person.py`)
  - Objects have 3 attributes
  - `name`: String
  - `mom`: Person (or `None`)
  - `dad`: Person (or `None`)
- Represents the "family tree"
  - Goes as far back as known
  - Attributes `mom` and `dad` are `None` if not known
- **Constructor**: `Person(n,m,d)`
  - Or `Person(n)` if no mom, dad

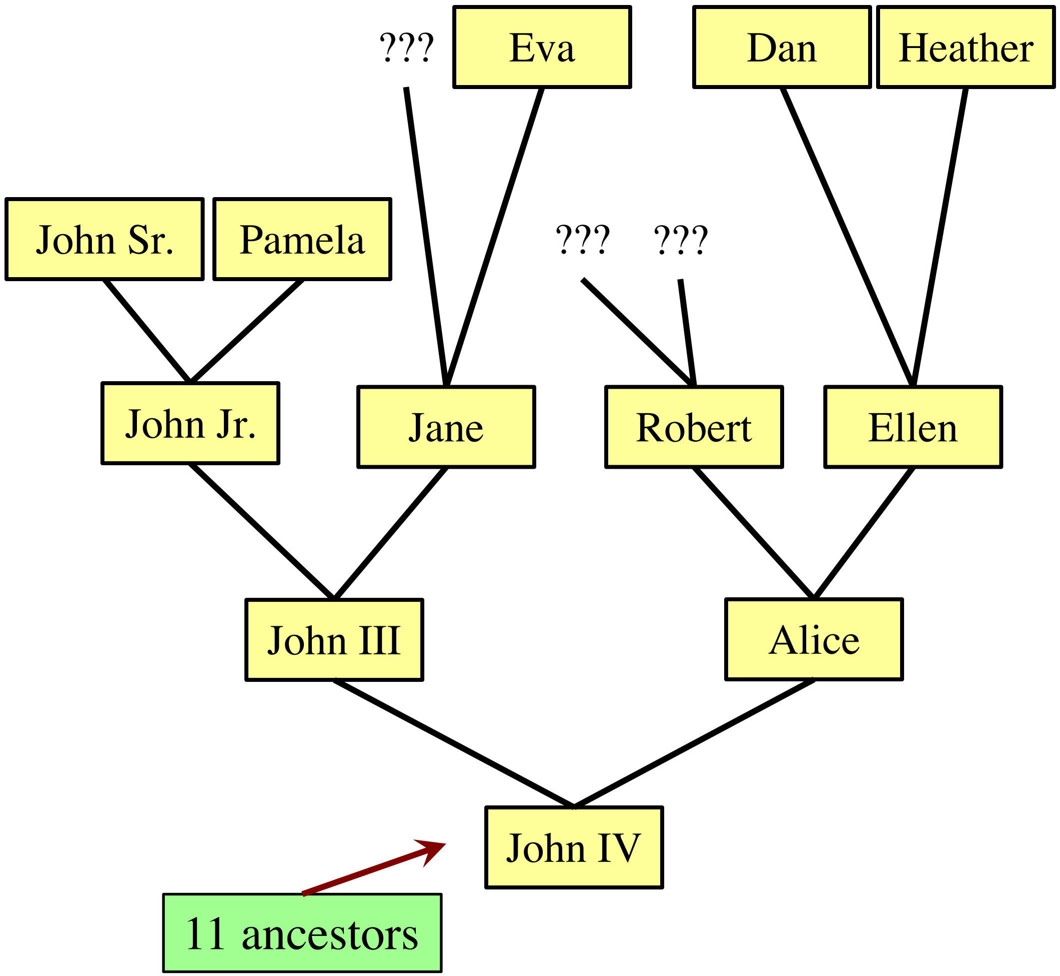


Recursion and Objects

```
def num_ancestors(p):
    """Returns: num of known ancestors
    Pre: p is a Person"""
    # 1. Handle small data.
    # No mom or dad (no ancestors)


    # 2. Break into two parts
    # Has mom or dad
    # Count ancestors of each one
    # (plus mom, dad themselves)


    # 3. Combine the result
```

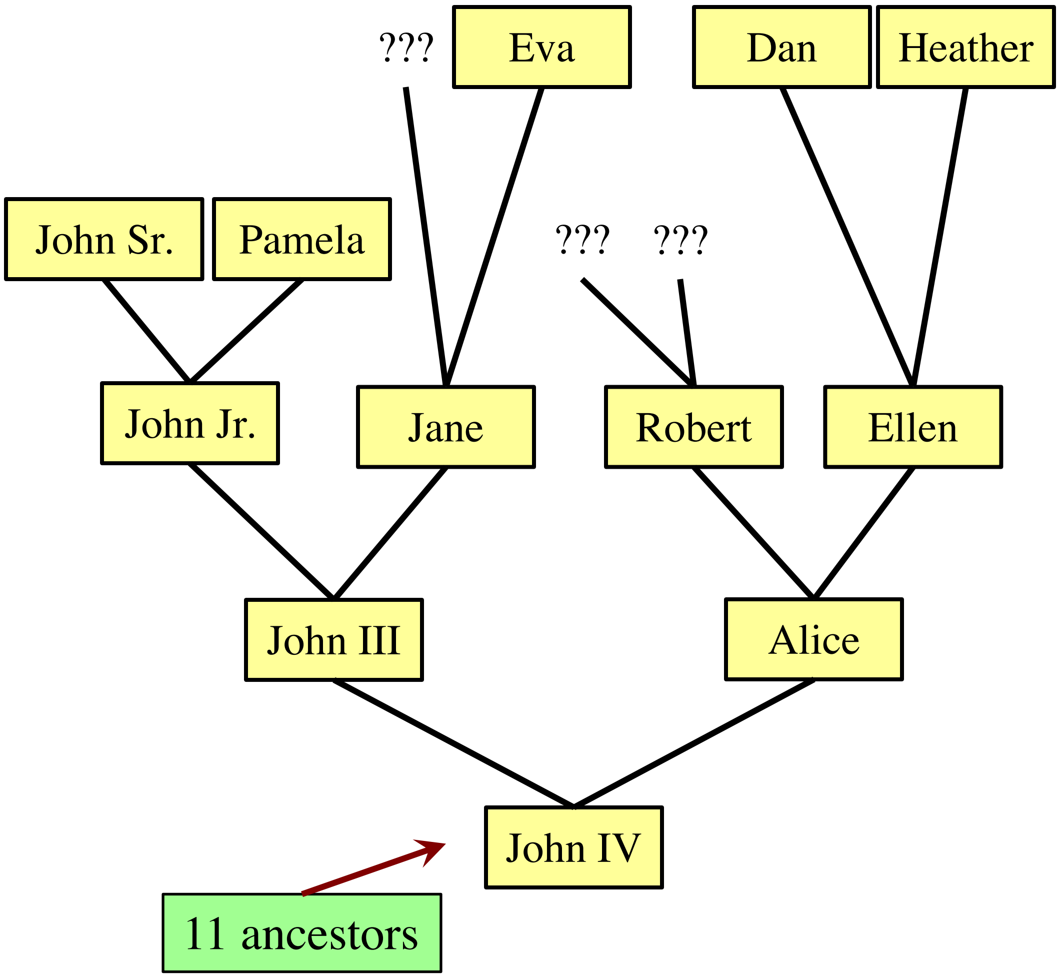


Recursion and Objects

```
def num_ancestors(p):
    """Returns: num of known ancestors
    Pre: p is a Person"""
    # 1. Handle small data.
    if p.mom == None and p.dad == None:
        return 0

    # 2. Break into two parts
    moms = 0
    if not p.mom == None:
        moms = 1+num_ancestors(p.mom)
    dads = 0
    if not p.dad== None:
        dads = 1+num_ancestors(p.dad)

    # 3. Combine the result
    return moms+dads
```



Is All Recursion Divide and Conquer?

- Divide and conquer implies two halves “equal”
  - Performing the same check on each half
  - With some optimization for small halves
- Sometimes we are given a **recursive definition**
  - Math formula to compute that is recursive
  - String definition to check that is recursive
  - Picture to draw that is recursive
  - **Example**: n! = n (n-1)!
- In that case, we are just implementing definition

Example: Palindromes

- String with ≥ 2 characters is a palindrome if:
  - its first and last characters are equal, and
  - the rest of the characters form a palindrome
- **Example:**

have to be the same

AMANAPLANACANALPANAMA

has to be a palindrome

- **Function to Implement:**

```
def ispalindrome(s):
    """Returns: True if s is a palindrome"""
```

Example: Palindromes

- String with ≥ 2 characters is a palindrome if:
  - its first and last characters are equal, and
  - the rest of the characters form a palindrome

Recursive Definition

```
def ispalindrome(s):
    """Returns: True if s is a palindrome"""
    if len(s) < 2:
        return True

    # Halves not the same; not divide and conquer
    ends = s[0] == s[-1]
    middle = ispalindrome(s[1:-1])
    return ends and middle
```

Base case

Recursive case

Recursive Functions and Helpers

```
def ispalindrome2(s):
    """Returns: True if s is a palindrome
    Case of characters is ignored."""
    if len(s) < 2:
        return True
    # Halves not the same; not divide and conquer
    ends = equals_ignore_case(s[0], s[-1])
    middle = ispalindrome(s[1:-1])
    return ends and middle
```

Recursive Functions and Helpers

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    """Returns: True if s is a palindrome
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Recursive Functions and Helpers

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    if len(s) < 2:
        return True
    # Halves not the same; not divide and conquer
    ends = equals_ignore_case(s[0], s[-1])
    middle = ispalindrome(s[1:-1])
    return ends and middle

def equals_ignore_case(a, b):
    """Returns: True if a and b are same ignoring case"""
    return a.upper() == b.upper()
```

Use helper functions!

- Pull out anything not part of the recursion
- Keeps your code simple and easy to follow

Example: More Palindromes

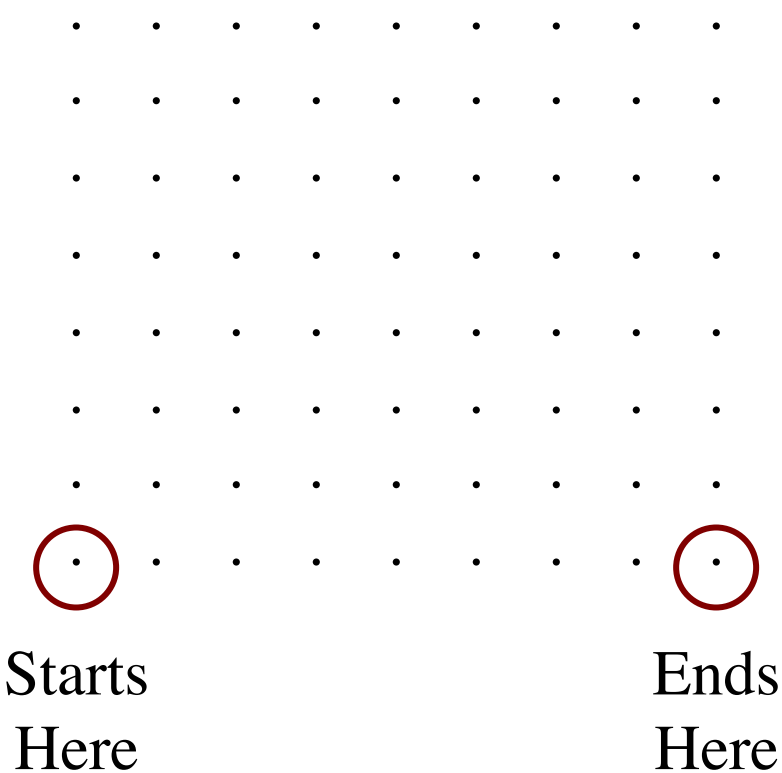
```
def ispalindrome3(s):
    """Returns: True if s is a palindrome
    Case of characters and non-letters ignored."""
    return ispalindrome2(depunct(s))


def depunct(s):
    """Returns: s with non-letters removed"""
    if s == '':
        return s
    # Combine left and right
    if s[0] in string.letters:
        return s[0]+depunct(s[1:])
    # Ignore left if it is not a letter
    return depunct(s[1:])
```

Use helper functions!

- Sometimes the helper is a recursive function
- Allows you break up problem in smaller parts

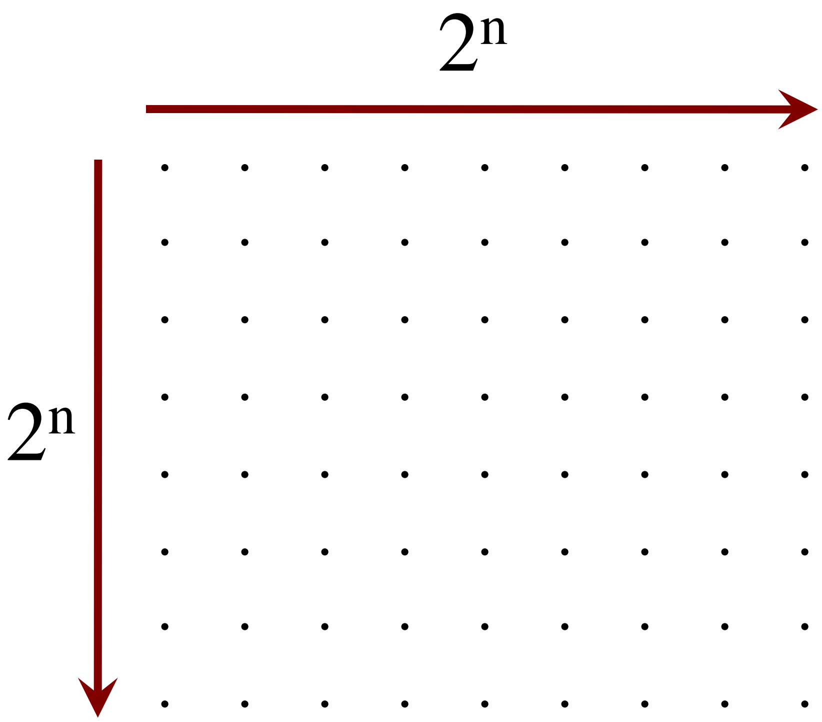
Example: Space Filling Curves

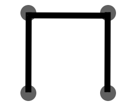


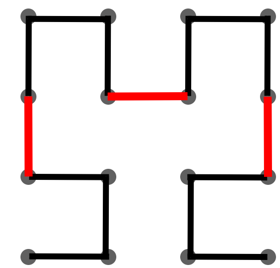
Challenge

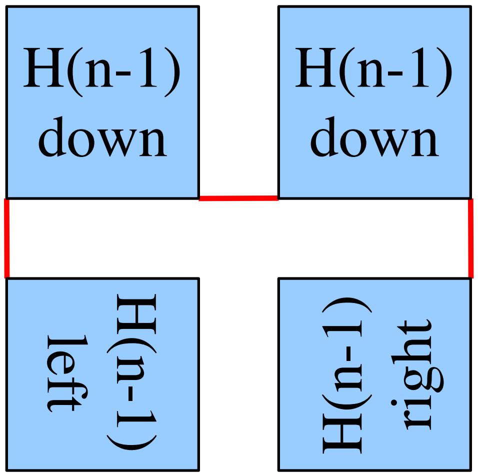
- Draw a curve that
  - Starts in the left corner
  - Ends in the right corner
  - Touches every grid point
  - Does not touch or cross itself anywhere
- Useful for analysis of 2-dimensional data

Hilbert's Space Filling Curve



Hilbert(1): 

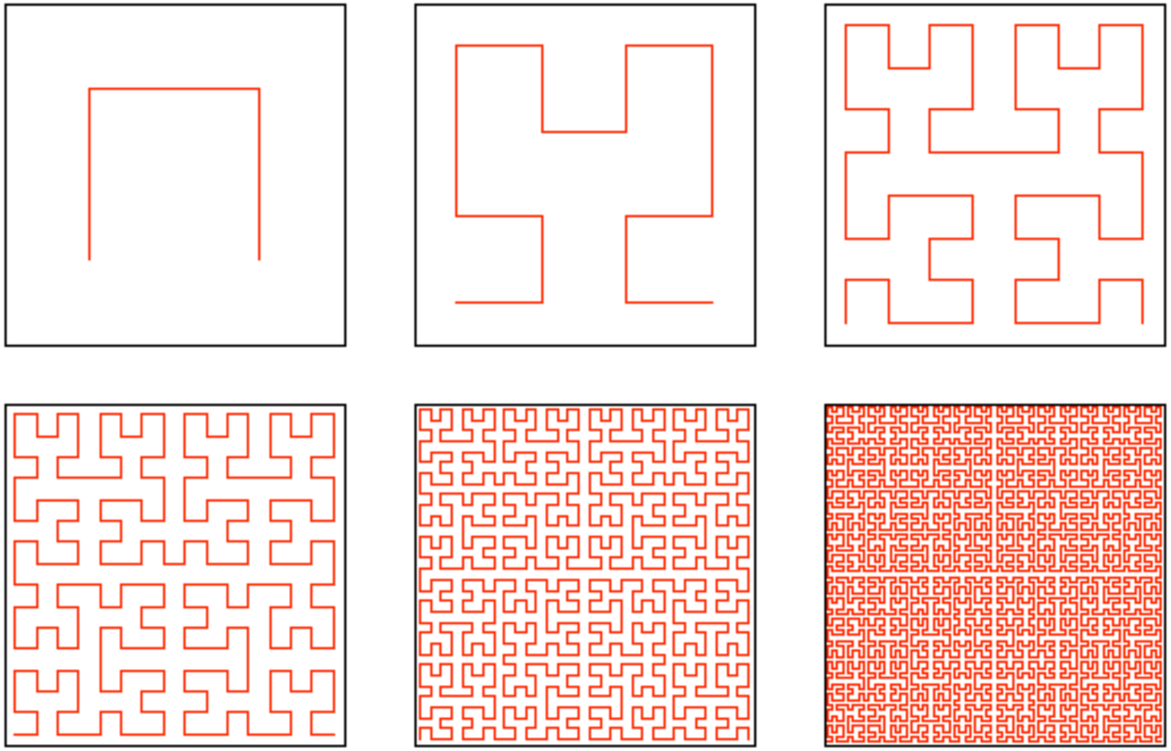
Hilbert(2): 

Hilbert(n): 

Hilbert's Space Filling Curve

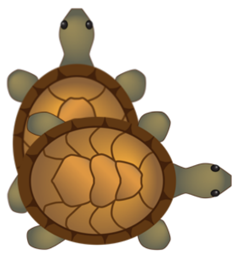
Basic Idea

- Given a box
- Draw $2^n \times 2^n$ grid in box
- Trace the curve
- As n goes to ∞, curve fills box

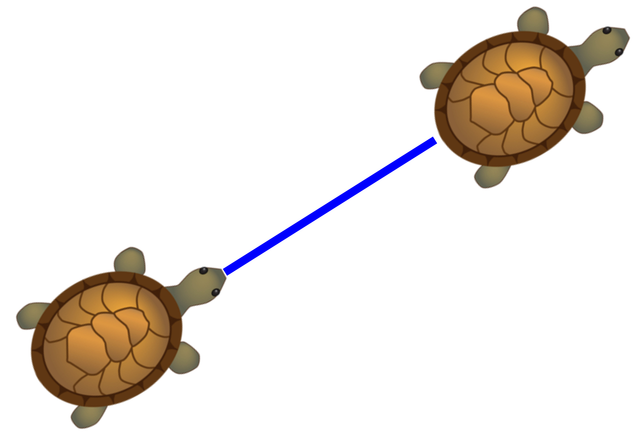


“Turtle” Graphics: Assignment A4

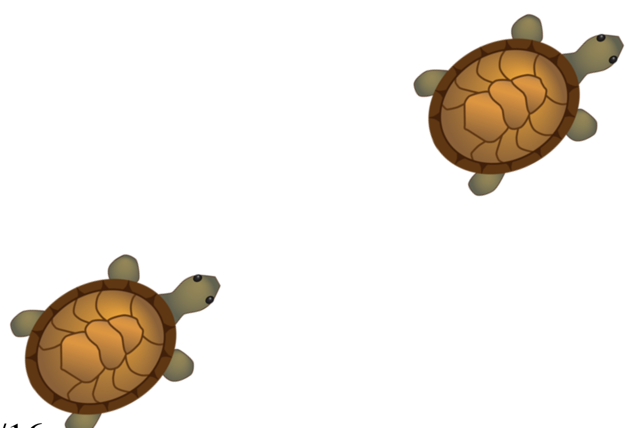
Turn



Draw Line



Move



Change Color