Quantum Predicates and Instruments

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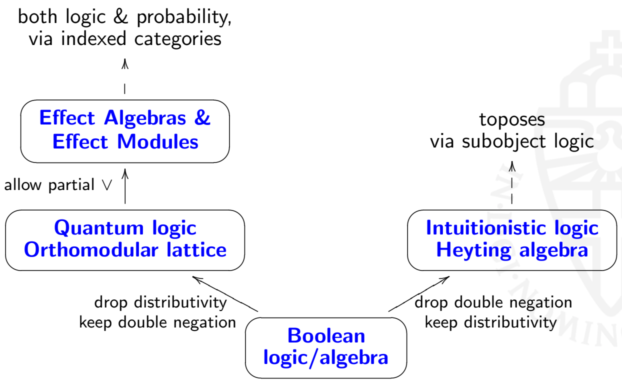
Outline

From Boolean to intuitionistic & quantum logic



Plan, overall & today

- This is part of an ongoing research project aiming at an axiomatisation of categorical quantum logic
  - For the wider picture, see `www.cs.ru.nl/B.Jacobs/TALKS/quantum-logic-6up.pdf`
- Today: focus on the basic part: predicates and instruments
- predicates form an effect algebra (or module)
  - they arise as maps of the form $X \to 1+1$
  - we show both examples and the general construction
  - and also their relation to states
- instruments do the associated measurement operation
  - measurement options and side-effects are made explicit
  - useful for guarded commands and dynamic logic

Main examples

- **Sets**, the category of sets and functions
- $\mathcal{K}\ell(\mathcal{D})$, the Kleisli category of the distribution monad $\mathcal{D}$
  - additionally $\mathcal{K}\ell(\mathcal{G})$, for the Giry monad $\mathcal{G}$
- Opposite categories $\mathbf{Rng}^{\mathrm{op}}$ or $\mathbf{Quant}^{\mathrm{op}}$ or $\mathbf{DistLat}^{\mathrm{op}}$, of rings, quantales, distributive lattices
- $(\mathbf{Cstar}_{\mathrm{UP}})^{\mathrm{op}}$ with of C^*-algebras, and variations
  - completely positive maps, W^*-algebras, subunital maps
  - the crucial, but trivial mental steps are:
    - not to use Hilbert spaces, but C^*-algebras
    - to work in the opposite category
    - to use unital positive (UP) maps instead of *-homomorphisms

Effect algebras, definition

Effect algebras axiomatise the unit interval $[0,1]$ with its (partial!) addition $+$ and "negation" $x \mapsto 1-x$.

A Partial Commutative Monoid (PCM) consists of a set M with zero $0 \in M$ and partial operation $\ovee\colon M \times M \to M$, which is suitably commutative and associative.

One writes $x \perp y$ if $x \ovee y$ is defined.

An effect algebra is a PCM in which each element x has a unique 'orthosuplement' $x^\perp$ with $x \ovee x^\perp = 1\ (= 0^\perp)$
Additionally, $x \perp 1 \Rightarrow x = 0$ must hold.

There is then a partial order, via $x \leq y$ iff $y = x \ovee z$, for some z.

Homomorphisms of effect algebras

Definition

A homomorphism of effect algebras $f\colon X \to Y$ satisfies:

- $f(1) = 1$
- if $x \perp x'$ then both $f(x) \perp f(x')$ and $f(x \ovee x') = f(x) \ovee f(x')$.

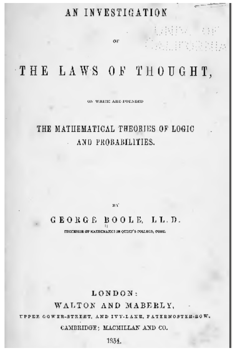
This yields a category **EA** of effect algebras.

Example:

- A probability measure yields a map $\Sigma_X \to [0,1]$ in **EA**.

Naturality of partiality

George Boole in 1854 thought of disjunction as a partial operation


AN INVESTIGATION
OF
THE LAWS OF THOUGHT,
ON WHICH ARE FOUNDED
THE MATHEMATICAL THEORIES OF LOGIC AND PROBABILITIES.
BY
GEORGE BOOLE, LL. D.
LONDON:
WALTON AND MABERLY,
CAMBRIDGE: MACMILLAN AND CO.
1854.

"Now those laws have been determined from the study of instances, in all of which it has been a necessary condition, that the classes or things added together in thought should be mutually exclusive. The expression $x+y$ seems indeed uninterpretable, unless it be assumed that the things represented by x and the things represented by y are entirely separate; that they embrace no individuals in common." (p.66)

Effect modules

Effect modules are effect algebras with a scalar multiplication, with scalars not from $\mathbb{R}$ or $\mathbb{C}$, but from $[0,1]$.
(Or more generally from an "effect monoid", ie. effect algebra with multiplication)

Definition

An effect module M is a effect algebra with an action $[0,1] \times M \to M$ that is a "bihomomorphism"

A map of effect modules is a map of effect algebras that commutes with scalar multiplication.

We get a category **EMod** $\hookrightarrow$ **EA**.

Effect modules, main examples

Probabilistic examples

- Fuzzy predicates $[0,1]^X$ on a set X, with scalar multiplication
$$r \cdot p \stackrel{\text{def}}{=} \lambda x \in X.\, r \cdot p(x).$$
- Measurable predicates $\mathrm{Hom}(X, [0,1])$, for a measurable space X, with the same scalar multiplication.

Quantum examples

- Effects $\mathcal{E}(H)$ on a Hilbert space: operators $A\colon H \to H$ satisfying $0 \leq A \leq I$, with scalar multiplication $(r, A) \mapsto rA$.
- Effects in a C^*-algebra A: positive elements below the unit:
$$[0,1]_A = \{a \in A \mid 0 \leq a \leq 1\}.$$

This one covers the previous three illustrations.

In a category with final object 1 and coproducts +

- An n-test is a map $X \to n \cdot 1 = 1 + \cdots + 1$
- a predicate is a 2-test, ie. a map $X \to 1 + 1 = 2$
  - notation: $\mathrm{Pred}(X) = \mathrm{Hom}(X, 2)$
- We get some logical structure for free:

$$1 = (1 \xrightarrow{\kappa_1} 1+1) \quad 0 = (1 \xrightarrow{\kappa_2} 1+1) \quad p^{\perp} = (X \xrightarrow{p} 1+1 \xrightarrow[\cong]{[\kappa_2,\kappa_1]} 1+1)$$

  Then $p^{\perp\perp} = p$, $0^{\perp} = 1$, $1^{\perp} = 0$.
- Predicates $1 \to 1+1$ on 1 will be called scalars
  - they carry a monoid structure $p \cdot q = [p, \kappa_2] \circ q$

Predicate examples: Boolean & fuzzy logic

- In **Sets**, maps $X \to 1 + 1 = 2$ correspond to subsets of X
  - an n-test $X \to n \cdot 1 = n$ corresponds to a disjoint cover of X
- In the Kleisli category $\mathcal{K}\ell(\mathcal{D})$, for a set X,

$$\frac{\frac{\text{Kleisli map } X \longrightarrow 2}{\text{function } X \longrightarrow \mathcal{D}(2) = [0,1]}}{\text{fuzzy predicate in } [0,1]^X}$$

- Similarly, in $\mathcal{K}\ell(\mathcal{G})$ predicates on a measurable space X are
  - measurable (fuzzy) functions $X \to [0,1]$
  - i.e. $[0,1]$-valued random/stochastic variables

The scalars in **Sets** are $\{0,1\}$, and in $\mathcal{K}\ell(\mathcal{D}), \mathcal{K}\ell(\mathcal{G})$ they are $[0,1]$.

Predicates in distributive lattices

- We work in the opposite category $\mathbf{DistLat}^{\mathrm{op}}$
  - more about this opposite later
- 1 and $+$ in $\mathbf{DistLat}^{\mathrm{op}}$ are initial and product in **DistLat**
- What is the initial distributive lattice? $2 = \{0, 1\}$
- For a distributive lattice L we get bijective correspondences:

$$\begin{array}{cl} L \longrightarrow 1+1 & \text{in } \mathbf{DistLat}^{\mathrm{op}} \\ \hline\hline 2 \times 2 \longrightarrow L & \text{in } \mathbf{DistLat} \\ \hline\hline \text{complementable elements } x \in L & \end{array}$$

where: x is complementable if there is a (necessarily unique) $x' \in L$ with $x \wedge x' = 0$ and $x \vee x' = 1$.

- $f\colon 2 \times 2 \to L$ gives $x = f(1,0)$ and $x' = f(0,1)$
- these complementable elements form a Boolean sublattice

Predicates in rings

- We play the same game in $\mathbf{Rng}^{\mathrm{op}}$ — with rings having a unit
  - the initial ring is: $\mathbb{Z}$
- We now have correspondences, for a ring R,

$$\begin{array}{cl} R \longrightarrow 1+1 & \text{in } \mathbf{Rng}^{\mathrm{op}} \\ \hline\hline \mathbb{Z} \times \mathbb{Z} \longrightarrow R & \text{in } \mathbf{Rng} \\ \hline\hline \text{idempotents } r \in R & \end{array}$$

Predicates in rings, continued

Theorem $\mathrm{Pred}(R) = \{r \in R \mid r^2 = r\}$ in a ring R is

1. an effect algebra
2. a Boolean algebra, in case R is commutative

(2) is well-known, (1) is new.

In $\mathrm{Pred}(R)$ one has:

- $r \perp s$ iff $rs = 0 = sr$, and in that case: $r \ovee s = r + s$
- orthocomplement $r^{\perp} = 1 - r$
- $r \leq s$ iff $rs = r = sr$, with 0 bottom and 1 top.

We get functors:

$$\begin{array}{ccc} \mathbf{Rng} & \hookleftarrow & \mathbf{CRng} \\ {\scriptstyle \mathrm{Pred}}\downarrow & & \downarrow{\scriptstyle \mathrm{Pred}} \\ \mathbf{EA} & \hookleftarrow & \mathbf{BA} \end{array}$$

Aside: tests and the Pierce decomposition

- An n-test $R \to n \cdot 1$ in $\mathbf{Rng}^{\mathrm{op}}$ corresponds to:
  - a ring homomorphism $f\colon \mathbb{Z}^n \to R$
  - n idempotents $e_i = f(|\,i\,\rangle) \in R$ with $e_1 + \cdots + e_n = 1$ and $e_i e_j = 0$ for $i \neq j$.
- Such an n-test is an essential ingredient of the Pierce decomposition of the ring R.

$$R \xrightarrow{\;\cong\;} \textstyle\bigoplus_{i,j} e_i R e_j$$

Predicates in quantales (with unit)

- The same game in $\mathbf{Quant}^{\mathrm{op}}$ yields:
  - predicates are idempotents with complements
  - they form an effect algebra again
- We get a similar diagram:

$$\begin{array}{ccc} \mathbf{Quant} & \hookleftarrow & \mathbf{CQuant} \\ {\scriptstyle \mathrm{Pred}}\downarrow & & \downarrow{\scriptstyle \mathrm{Pred}} \\ \mathbf{EA} & \hookleftarrow & \mathbf{BA} \end{array}$$

$C*$-algebras (with unit)

The complex numbers $\mathbb{C}$ are initial in $\mathbf{Cstar}_{\mathrm{UP}}$, so final in $(\mathbf{Cstar}_{\mathrm{UP}})^{\mathrm{op}}$. Hence, $1 + 1 = \mathbb{C} \oplus \mathbb{C} = \mathbb{C}^2$, so:

$$\begin{array}{cl} A \longrightarrow 2 & \text{in } (\mathbf{Cstar}_{\mathrm{UP}})^{\mathrm{op}} \\ \hline\hline \mathbb{C}^2 \longrightarrow A & \text{in } \mathbf{Cstar}_{\mathrm{UP}} \\ \hline\hline \text{effect in } [0,1]_A \subseteq A & \end{array}$$

This $A \mapsto [0,1]_A$ is a full&faithful functor $\mathbf{Cstar}_{\mathrm{UP}} \to \mathbf{EMod}$.

Why all these opposite categories?

In non-deterministic program semantics there are bijective correspondences:

$$\frac{\dfrac{X \xrightarrow{s} \mathcal{P}(Y)}{\mathcal{P}(X) \longrightarrow \mathcal{P}(Y)} \ \bigvee\text{-preserving}}{\mathcal{P}(Y) \xrightarrow[\mathrm{wp}(s)]{} \mathcal{P}(X)} \ \bigwedge\text{-preserving}$$

The opposite $(-)^{\mathrm{op}}$ arises when we look at maps as predicate transformers, working backwards.

The underlying result

Proposition

Assuming coproducts in **B** are "nice",

1. each $\mathrm{Pred}(X)$ is an effect module over the scalars $\mathrm{Pred}(1)$
2. this yields a functor (or "indexed category")

$$\mathbf{B} \xrightarrow{\mathrm{Pred}} \mathbf{EMod}^{\mathrm{op}}$$

States

Definition

A state on object X is a map $\omega\colon 1 \to X$.
Write $\mathrm{Stat}(X) = \mathrm{Hom}(1, X)$.

For a predicate $p\colon X \to 1+1$ define the validity probability via an abstract version of the Born rule:

$$\omega \models p \quad \stackrel{\text{def}}{=} \quad p \circ \omega \colon 1 \to 1+1$$

Lemma $\mathrm{Stat}(X)$ is a convex sets, closed under convex sums with scalars adding to 1.

States and validity examples

- In **Sets**, states are elements (and predicates subsets), and:
$$x \models p \quad = \quad p(x) \ \in \ \{0,1\}$$
- In $\mathcal{K}\ell(\mathcal{D})$, states are distributions $\varphi \in \mathcal{D}(X)$, and:
$$\varphi \models p \quad = \quad \sum_x p(x)\cdot\varphi(x) \ \in \ [0,1]$$
- In $\mathcal{K}\ell(\mathcal{G})$, states are probability measures $\phi \in \mathcal{G}(X)$, and:
$$\phi \models p \quad = \quad \int p \,\mathrm{d}\phi(x) \ \in \ [0,1]$$
- In $(\mathbf{Cstar}_{\mathrm{UP}})^{\mathrm{op}}$, states are positive unital maps $A \to \mathbb{C}$, and:
$$\omega \models p \quad = \quad \omega(p) \ \in \ [0,1]$$

States, programs, predicates

We read maps in **B** in the following manner

$$\begin{cases} \text{states} & \omega\colon 1 \longrightarrow X \\ \text{programs} & f\colon X \longrightarrow Y \\ \text{predicates} & q\colon Y \to 1+1 \end{cases}$$

Each $f\colon X \to Y$ yields two "transformer" maps:

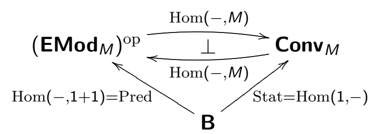
$$\begin{cases} \text{state transformer} & f_* = f \circ (-)\colon \mathrm{Stat}(X) \longrightarrow \mathrm{Stat}(Y) \\ \text{predicate transformer} & f^* = (-) \circ f = \mathrm{wp}(f)\colon \mathrm{Pred}(Y) \longrightarrow \mathrm{Pred}(X) \end{cases}$$

There is the "Galois" equation for the validity probability:

$$(f_*(\omega) \models q) \ = \ (\omega \models f^*(q)) \ = \ (1 \xrightarrow{\omega} X \xrightarrow{f} Y \xrightarrow{q} 1+1).$$

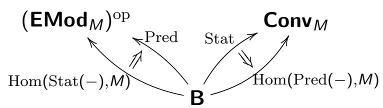
Summary

If **B** has nice coproducts, there is a state-and-effect triangle:



where $M = \mathrm{Pred}(1) = \mathrm{Stat}(2)$

Validity $\models$ yields two natural transformations:



Tests and instruments

We also require:

For each n-test $p\colon X \to n\cdot 1 = 1 + \cdots + 1$ there is an instrument $instr_p\colon X \to n\cdot X = X + \cdots + X$ in $\mathbf{B}$ satisfying:

$$X \xrightarrow{p} n\cdot 1, \qquad X \xrightarrow{instr_p} n\cdot X \xrightarrow{n\cdot !} n\cdot 1$$

satisfying some "coherence" conditions.

These instruments capture both:

- the different outcome options, via the coproducts $X + \cdots + X$
- the side-effect (aka. observer effect) of a test p is:

$$X \xrightarrow{instr_p} X + \cdots + X \xrightarrow{\nabla = [\mathrm{id},\ldots,\mathrm{id}]} X$$

If this map is the identity, we call p side-effect free.

Instruments: examples

- An n-test in **Sets** consists of disjoint subsets $P_i \subseteq X$ that cover X, and gives $instr_p\colon X \to n\cdot X$ by:

$$instr_p(x) = \kappa_i x \quad \text{iff} \quad x \in P_i.$$

- An n-test in $\mathcal{K}\ell(\mathcal{D})$ consists of n predicates $p_i\colon X \to [0,1]$ that sum to 1, so we get map $instr_p\colon X \to \mathcal{D}(n\cdot X)$ by:

$$instr_p(x) = p_1(x)\kappa_1 x + \cdots + p_n(x)\kappa_n x$$

- In $\mathcal{K}\ell(\mathcal{G})$ we get $instr_p\colon X \to \mathcal{G}(n\cdot X)$, with for $M \in \Sigma_X$,

$$instr_p(x)(\kappa_i M) = p(x)(i)\cdot \mathbf{1}_M(x)$$

- An n-test in a C^*-algebra A consist of effects $e_i \in [0,1]_A$ summing to 1, and gives $instr_{\vec{e}}\colon A \to n\cdot A$ in $(\mathbf{Cstar}_{\mathrm{UP}})^{\mathrm{op}}$, so $instr_{\vec{e}}\colon A^n \to A$ in $\mathbf{Cstar}_{\mathrm{UP}}$, with:

$$instr_{\vec{e}}(x_1,\ldots,x_n) = \sqrt{e_1}\cdot x_1\cdot\sqrt{e_1} + \cdots + \sqrt{e_n}\cdot x_n\cdot\sqrt{e_n}$$

Side-effect freeness

- Tests/predicates are side-effect-free in
  - **Sets**
  - in $\mathcal{K}\ell(\mathcal{D})$ and $\mathcal{K}\ell(\mathcal{G})$
  - in *commutative* C^*-algebras
- In fact, one can prove: a C^*-algebra is commutative iff all its effects are side-effect-free.

Quantum instruments

- Instruments $X \to X + \cdots + X$ distinguish
  - different outcomes, via coproduct options
  - side-effects (aka. observer effect)
- The notion of instrument goes back to
  - Davies & Lewis, *An Operational Approach to Quantum Probability*, CMP 1970
  - Ozawa, *Quantum measuring processes of continuous observables*, JMP 1984
  - See also: Heinosaari & Zimon book, 2012
- We give a categorical formalisation of discrete instruments

The Davies/Lewis and Ozawa formulation

- C^*-algebraically, an instrument on A is a measurable-set-indexed collection of subunital completely positive maps:

$$\Big(A \xrightarrow{f_M} A\Big)_{M\in\Sigma}$$

  such that:
  - $f_{\bigcup_i M_i} = \sum_i f_{M_i}$, for a pairwise disjoint collection $M_i \in \Sigma$
  - f_X is unital, where X is the underlying space of $\Sigma \subseteq \mathcal{P}(X)$.
- Here: $instr_{\vec{e}}\colon A^n \to A$ via $instr_{\vec{e}}(a_1,\ldots,a_n) = \sum_i \sqrt{e_i}\cdot a_i\cdot\sqrt{e_i}$
  - take the discrete measurable space n, with $\Sigma = \mathcal{P}(n)$
  - define for $M \in \Sigma$, the map $f_M\colon A \to A$ by:

$$f_M(a) = \textstyle\sum_{i\in M}\sqrt{e_i}\cdot a\cdot\sqrt{e_i}$$

  - the additivity condition holds by construction
  - and: $f_n(1) = instr_{\vec{e}}(1) = \sum_i e_i = 1$.

Final remarks

- Effect algebras/modules arise naturally
  - not only in examples: fuzzy predicates, idempotents in a ring, effects in C^*-algebras
  - but also from basic categorical structure
- States-and-effect triangles capture basics of program semantics
  - duality between state- and predicate-transformations
- Axiomatisation of (categorical) quantum logic is well underway, via several basic assumptions (paper soon finished)
- A corresponding calculus of types, terms and formulas has been developed by Robin Adams (QPL'14)