Theory Qualifier

January 15, 1998

Instructions

Please read these instructions and all questions C A R E F U L L Y before proceeding.

You will have $2\frac{1}{2}$ hours. The exam is closed-book. All questions are of equal weight. You must attempt to answer all of them to earn an unconditional pass. All algorithms should be accompanied by a proof of correctness and complexity analysis unless otherwise directed.

The questions will be graded according to the following criteria in order of importance:

1. correctness and completeness
2. clarity, precision, and conciseness
3. optimality of result

Partial credit will be awarded where appropriate, so be sure to show all work. Unclear or irrelevant arguments will be penalized.

Good luck!

1. Given a directed acyclic graph G and two nodes s and t, we would like to compute the number of directed paths from s to t in G. Give an efficient algorithm for this problem. Full credit will be given for solutions that run in linear time (you may assume constant-time arithmetic). Exponential-time solutions will not be worth much.

2. Design a data structure to represent undirected graphs that can do the following.
   - Given two nodes, test whether there is an edge between them in $O(1)$ time.
   - Insert a new edge connecting two given nodes in $O(1)$ time.
   - Delete a given edge from the graph in $O(1)$ time.
   - Given a node i, list all the nodes adjacent to i in time $O(\text{degree of } i)$.

3. (a) Show that if we had a procedure that could decide whether a given TM M accepts exactly two strings, then we could use it to decide whether a given TM N accepts exactly one string.

   (b) Show that if we had a procedure that could decide whether a given TM M accepts exactly one string, then we could use it to decide whether a given TM N accepts exactly two strings.

4. An undirected graph is a *near clique* if it is a complete graph except for one missing edge; that is, if it has $\binom{n}{2} - 1$ edges, where n is the number of vertices. Prove that the following problem is NP-complete: given an undirected graph G and a number k, does G contain an induced near clique on k vertices?

End of Exam