Theory Qualifier
January 18, 1995

Instructions

Please read these instructions and all questions C A R E F U L L Y before proceeding.

You will have $2\frac{1}{2}$ hours. The exam is closed-book. All questions are of equal weight. You must attempt to answer all of them to earn an unconditional pass. All algorithms should be accompanied by a proof of correctness and complexity analysis unless otherwise directed.

The questions will be graded according to the following criteria in order of importance:

1. correctness and completeness
2. clarity, precision, and conciseness
3. optimality of result

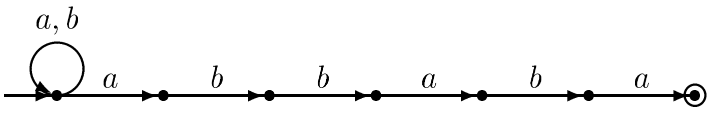
Partial credit will be awarded where appropriate, so be sure to show all work. Unclear or irrelevant arguments will be penalized.

Good luck!

1. (Automata and formal languages) Let $\Sigma = \{a, b\}$. For any $x \in \Sigma^*$, define

$$A_{-x} \;=\; \{ux \mid u \in \Sigma^*\} \; ,$$

the set of strings ending with x. The set A_{-x} is accepted by a nondeterministic finite automaton with $|x|+1$ states. For example, here is a nondeterministic finite automaton for $A_{-abbaba}$:



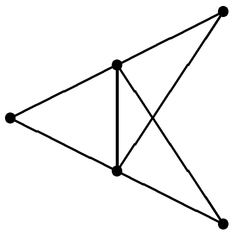
(a) Draw the minimal *deterministic* finite automaton for $A_{-abbaba}$.

(b) Argue that for any x, the minimal deterministic finite automaton for A_{-x} has exactly $|x|+1$ states. (*Hint.* Consider the relation

$$y \equiv z \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad \text{overlap}\,(y, x) = \text{overlap}\,(z, x) \; ,$$

where overlap (y, x) is the longest string that is both a suffix of y and a prefix of x.)

2. (Complexity, completeness and reducibility) Show that the following problem is *NP*-complete: given an undirected graph of maximum vertex degree 4, does it have a 3-coloring? You may use the fact that 3-coloring is *NP*-complete for undirected graphs without the degree restriction. *Hint*:



3. (Computability and undecidability) Consider a simple imperative programming language with the following programming constructs:

$x := e$	assignment
if $x = y$ **then** p **else** q	conditional
while $x = y$ **do** p	while loop
$p; q$	sequential composition
print x **and halt**	halt with output

Program variables $x, y, \ldots$ range over $\mathbb{N}$. The expression e in the assignment statement can be any arithmetic expression involving $+$, $\cdot$, variables, or numeric constants. The test $x = y$ in the conditional and while loop can be replaced by $x \neq y$, $x \leq y$, or $x < y$. Two programs are *equivalent* if for any initial values of the variables, one program halts iff the other does, and whenever they both halt, then they print the same value.

Say whether the following problems are decidable or undecidable. Justify your answers.

(a) Given a program, does there exists an equivalent program with at most one while loop?

(b) Given a program, does there exists an equivalent program with no while loops?

4. (Design and analysis of algorithms) A well-known *NP*-complete problem is the *set cover problem*: Given a family S of subsets of X, find a subfamily $S' \subseteq S$ of minimum cardinality that covers X in the sense that $X = \bigcup S'$. Show that the problem is solvable in time $O(n \log n)$ if X and the elements of S are subintervals $[u, v]$ of a discrete linear order in which comparisons take unit time.

END OF EXAM