Theory Qualifier
January 15, 1991

Instructions

Please read these instructions and all questions C A R E F U L L Y before proceeding.

You will have $2\frac{1}{2}$ hours. The exam is closed-book. All questions are of equal weight. You must attempt to answer all of them to earn an unconditional pass. All algorithms should be accompanied by a proof of correctness and complexity analysis unless otherwise directed.

The questions will be graded according to the following criteria in order of importance:

1. correctness and completeness
2. clarity, precision, and conciseness
3. optimality of result

Partial credit will be awarded where appropriate, so be sure to show all work. Unclear or irrelevant arguments will be penalized.

Good luck!

1. Let Σ be a finite alphabet. For $x \in \Sigma^*$ and $a \in \Sigma$, let $\#_a(x)$ denote the number of occurrences of a in x. For a set $A \subseteq \Sigma^*$, define

$$\begin{aligned} \textbf{cycle}\, A &= \{yx \mid xy \in A\} \\ \textbf{shuffle}\, A &= \{y \mid \exists x \in A\ \forall a \in \Sigma\ \#_a(x) = \#_a(y)\}\ . \end{aligned}$$

For example, if $\Sigma = \{a, b, c\}$ and $A = \{aaabc\}$, then

$$\begin{aligned} \textbf{cycle}\, A &= \{aaabc, aabca, abcaa, bcaaa, caaab\} \\ \textbf{shuffle}\, A &= \{aaabc, aabca, abcaa, bcaaa, caaab, \\ &\qquad aabac, abaca, bacaa, acaab, caaba, \\ &\qquad abaac, baaca, aacab, acaba, cabaa, \\ &\qquad baaac, aaacb, aacba, acbaa, cbaaa\}\ . \end{aligned}$$

Which of the following propositions are true and which are false? Give proofs for (i), (ii), and (iv). (*Extra credit.* Give a proof for (iii).)

(i) For all $A \subseteq \Sigma^*$, if A is regular, then so is **cycle** A.

(ii) For all $A \subseteq \Sigma^*$, if A is regular, then so is **shuffle** A.

(iii) For all $A \subseteq \Sigma^*$, if A is context-free, then so is **cycle** A.

(iv) For all $A \subseteq \Sigma^*$, if A is context-free, then so is **shuffle** A.

2. Are the following problems decidable? Give proofs.

(i) Does a given Turing machine M_i take at least i steps on some input?

(ii) ... all inputs?

3. Let X be a finite set and $\mathcal{A}$ a family of subsets of X. A *system of distinct representatives* for $(X, \mathcal{A})$ is a one-to-one mapping $f : \mathcal{A} \to X$ such that $f(A) \in A$ for all $A \in \mathcal{A}$. Give an efficient algorithm for determining whether a given $(X, \mathcal{A})$ has a system of distinct representatives and for finding one if so.

MORE—PLEASE GO ON TO THE NEXT PAGE

4. The following algorithm, known as *Prim's algorithm*, produces a minimum-weight spanning tree T in a connected undirected graph with edge weights. Initially, we choose an arbitrary vertex and let T be the tree consisting of that vertex and no edges. We then repeat the following step $n-1$ times: find an edge of minimum weight with exactly one endpoint in T and include that edge in T.

   (i) Argue that Prim's algorithm is correct.

   (ii) Give an implementation that runs in time $O(m + n \log n)$.

5. A world-famous professor from New York is planning a lecture tour to various countries. Banks in each country offer two exchange rates for each foreign currency, one for buying and one for selling, and travelers generally lose money with each exchange of currency. The professor knows in advance exactly how much of each currency she will need and wishes to do whatever she can to minimize her losses to currency exchange.

   (i) Suppose that she would like to carry only the currency of the country she is visiting in order to avoid confusion, and therefore plans to exchange all the currency she is holding at each border. Assuming that the exchange rates will be stable during her trip, she wishes to arrange her itinerary so as to visit each country exactly once and minimize her losses to currency exchange. Show that this problem is *NP*-complete.

   (ii) Suppose that she changes her mind about holding only one currency at a time and is willing to buy all the currency she will need in advance. All the banks have branches in New York, and she is willing to visit each one as many times as necessary to get the best deal. Show that this version of the problem is solvable in polynomial time.

END OF EXAM