Theory Qualifier
January 16, 1990

Instructions

Please read these instructions and all questions C A R E F U L L Y before proceeding.

You will have $2\frac{1}{2}$ hours. The exam is closed-book. All questions are of equal weight. You must attempt to answer all of them to earn an unconditional pass. All algorithms should be accompanied by a proof of correctness and complexity analysis unless otherwise directed.

The questions will be graded according to the following criteria in order of importance:

1. correctness and completeness
2. clarity, precision, and conciseness
3. optimality of result

Partial credit will be awarded where appropriate, so be sure to show all work. Unclear or irrelevant arguments will be penalized.

GOOD LUCK!

1. For a set A of natural numbers, define

$$\begin{aligned}
\textbf{binary}\,(A) &= \{\text{binary representations of numbers in } A\} \quad\subseteq\quad (0+1)^* \\
\textbf{unary}\,(A) &= \{0^n \mid n \in A\} \quad\subseteq\quad 0^*\ .
\end{aligned}$$

For example, if $A = \{2, 3, 5\}$, then

$$\begin{aligned}
\textbf{unary}\,(A) &= \{00, 000, 00000\} \\
\textbf{binary}\,(A) &= \{10, 11, 101\}\ .
\end{aligned}$$

Consider the following two propositions:

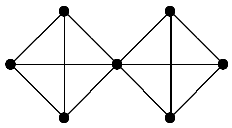
(i) For all A, if **binary** (A) is regular, then so is **unary** (A).

(ii) For all A, if **unary** (A) is regular, then so is **binary** (A).

One of (i), (ii) is true and the other is false. Which is which? Give proofs.

2. Let G be a connected undirected graph. We say G is *k-connected* if the deletion of any $k-1$ vertices leaves G connected. Give a polynomial-time algorithm (ideally, $O(k^2mn)$) for testing k-connectivity, and for finding a set of $k-1$ disconnecting vertices if G is not k-connected. (*Hint.* Use *Menger's Theorem*, which states that G is k-connected if and only if any pair of vertices is connected by at least k vertex-disjoint paths, then use max flow. You need not prove Menger's Theorem.)

3. The *bowtie graph* B_n consists of two copies of the complete graph K_n with a vertex in one copy identified with a vertex in the other. Here is B_4:



Show that the following problem is NP-complete: given an undirected graph G and an integer k, does G contain a subgraph isomorphic to B_k?

(more—please go on to next page)

4. Consider the following context-sensitive productions.

$$\begin{array}{rcll} S & \to & bSb & \\ S & \to & AcA & \\ Ab & \to & A & \\ Ab & \to & b & \\ bA & \to & b & \\ bA & \to & A & (*) \end{array}$$

Let G be the grammar given by all the rules other than (*), and G^* the grammar given by all the rules including (*). One of $\mathcal{L}(G)$ and $\mathcal{L}(G^*)$ is regular, and the other is context-free but not regular. Which is which, and why?

5. A *ray automaton* consists of an infinite number of deterministic finite-state automata $A_0, A_1, A_2, \ldots$ arranged in a line. The automata all have the same set of states Q, and all have the same transition function δ except A_0, which has a different transition function δ_0 since it has no left neighbor. They all start simultaneously in their initial state q_0 and execute synchronously. In each step, each A_i moves to a new state, depending on its own current state and the current states of its immediate left and right neighbors, according to its transition function. The ray automaton is said to *halt* if A_0 ever enters a distinguished final state q_f.

   (a) Give a rigorous formal definition of ray automata, including a definition of execution and halting. Your definition should begin: "A *ray automaton* is a tuple

   $$\mathcal{A} \;=\; (Q,\ q_0,\ q_f,\ \delta_0,\ \delta)$$

   where Q is a finite set of of *states*,..."

   (b) Is the halting problem for ray automata decidable? Why or why not?

   (c) ...semidecidable? Why or why not?

End of Exam