Theory Qualifier

January 17, 1989

Instructions

Please read these instructions and all questions C A R E F U L L Y before proceeding.

You will have $2\frac{1}{2}$ hours. The exam is closed-book. All questions are of equal weight. You must attempt to answer all of them to earn an unconditional pass. All algorithms should be accompanied by a proof of correctness and complexity analysis unless otherwise directed.

The questions will be graded according to the following criteria in order of importance:

1. correctness and completeness
2. clarity, precision, and conciseness
3. optimality of result

Partial credit will be awarded where appropriate, so be sure to show all work. Unclear or irrelevant arguments will be penalized.

Good luck!

T1. Given a bipartite graph $G = (U, V, E)$, say that $A \subseteq U$ is *deficient* if $|A| > |N(A)|$, where $N(A)$ denotes the set of neighbors of A; i.e.,

$$N(A) = \{v \in V \mid \exists u \in A \ (u, v) \in E\} .$$

(a) Give a polynomial-time algorithm for finding a deficient set if one exists. (*Hint.* Use Hall's Theorem. Grow a Hungarian tree.)

(b) Show that your algorithm yields a minimal deficient set.

T2. Give a linear-time algorithm for marking a given directed tree so that subsequent queries of the form, "Is vertex u an ancestor of vertex v?" can be answered in constant time.

T3. Given a satisfiable Boolean formula in 2CNF, show that a satisfying assignment can be found in

(a) linear time;

(b) NC.

T4. For $L \subseteq \Sigma^*$, define

$$L_\exists = \{x \mid \exists y \in \Sigma^* \ xy \in L\} .$$

If L is context free, is $L_\exists$ necessarily context free? Prove your answer.

Bonus. Answer the same question for

$$L_\forall = \{x \mid \forall y \in \Sigma^* \ xy \in L\} .$$

T5. In the following, let M_i, L_i, G_i, and R_i denote, respectively, the i^{th} Turing machine, linear bounded automaton, context-free grammar, and regular grammar in some standard enumeration. Tell which of the following are

(a) recursive,

(b) r.e. but not recursive,

(c) not r.e.

Give short explanations for (d), (e), (k), and (l).

(a) $\{M_i \mid L(M_i) \text{ is infinite}\}$

(b) $\{M_i \mid L(M_i) \neq \Sigma^*\}$

(c) $\{M_i \mid L(M_i) = \Sigma^*\}$

(d) $\{M_i \mid |L(M_i)| \geq 10\}$

(e) $\{M_i \mid |L(M_i)| \leq 10\}$

(f) $\{M_i \mid |L(M_i)| = 10\}$

(g) $\{L_i \mid L(L_i) \text{ is finite}\}$

(h) $\{L_i \mid L(L_i) = \Sigma^*\}$

(i) $\{L_i \mid L(L_i) \neq \Sigma^*\}$

(j) $\{G_i \mid L(G_i) \text{ is infinite}\}$

(k) $\{G_i \mid L(G_i) \neq \Sigma^*\}$

(l) $\{G_i \mid L(G_i) = \Sigma^*\}$

(m) $\{G_i \mid |L(G_i)| \geq 50\}$

(n) $\{(G_i, R_j) \mid L(G_i) \subseteq L(R_j)\}$

(o) $\{(G_i, R_j) \mid L(R_j) \subseteq L(G_i)\}$

End of Exam