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1 Multicut

We will now study an example of randomized rounding, where the geometry of the solution space is used in constructing the approximation algorithm.

The multicut probem has as input an undirected graph $G = (V, E)$, nonnegative costs associated with each edge (i.e. $c : E \to \mathbb{Q}^+$) and k terminal pairs (s_1, t_1), (s_2, t_2), $\ldots$, (s_k, t_k). A multicut is a subset of edges, F, such that removing these edges disconnects all s_i from t_i, for all i. The objective is to find a multicut of minimal total capacity. It is shown in [1] that it is NP-hard to find a multicut for $k \geq 3$.

Note that the capacity of any subset of edges, such that removing these edges disconnects all s_i from t_i, for all i, will be an upperbound on the maximal fractional multicommodity flow, so in particular the capacity of a multicut will be an upperbound.

Let's look at an IP-formulation of the multicut problem. Define

$$d(ij) = \begin{cases} 1 & \text{if } ij \in F \\ 0 & \text{otherwise} \end{cases}$$

for all $ij \in E$. Also define $\mathcal{P}_\ell$ as the set of all paths between s_ℓ and t_ℓ.

IP-formulation:

$$\begin{aligned} &\min \sum_{ij \in E} c(ij) d(ij) \\ &\text{s.t. } \sum_{ij \in P} d(ij) \geq 1 \text{ for all } P \in \mathcal{P}_\ell,\ \ell = 1, 2, \ldots, k \\ &d(ij) \in \{0, 1\} \end{aligned}$$

Note (1) that we can put all edges that are not present in E in, as long as we give them capacity 0, without changing the optimal solution and the optimal value of the problem — so let's do that.

We relax the constraints $d(ij) \in \{0, 1\}$ to get the LP-relaxation:

$$\begin{aligned} &\min \sum_{ij} c(ij) d(ij) \\ &\text{s.t. } \sum_{ij \in P} d(ij) \geq 1 \text{ for all } P \in \mathcal{P}_\ell,\ \ell = 1, 2, \ldots, k \\ &d(ij) \geq 0. \end{aligned}$$

Note (2) that the number of constraints of this problem is exponential. We can solve the LP in polynomial time, however, using the ellipsoid method [4], since we have a separation oracle (for more on this, see e.g. [3]): namely a shortest path algorithm.

The dual of the LP is:

$$\begin{aligned}
&\max \sum_{\ell, P \in \mathcal{P}_\ell} f_P^\ell \\
&\text{s.t.} \sum_{\ell,\, P \in \mathcal{P}_\ell \,\text{s.t.}\, ij \in P} f_P^\ell \leq c(ij) \text{ for all } ij \\
&f_P^\ell \geq 0.
\end{aligned}$$

Note that this is a max flow problem.

Claim 1. There exists an optimal solution to the LP-relaxation which is a semi-metric.[1]

Proof: Take any feasible solution d. The shortest path metric d' is also feasible and the objective value is not larger. □

We can therefore restrict our attention to (optimal) solutions, that are semi-metrics — this property is the "geometry of the solution space" we will exploit in constructing an integral solution.

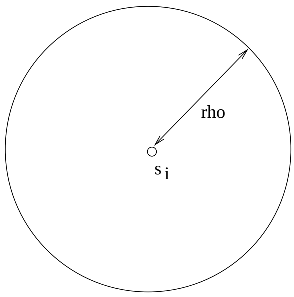
Our goal for the remainder of this lecture is to prove the following theorem.

Theorem 2. For every feasible solution d, there is a multicut F, such that

$$\sum_{ij \in F} c_{ij} \leq \alpha \sum_{ij} c(ij) d(ij),$$

where α will turn out to be $4 \ln k$.

We will create a solution using *ball cuts*:



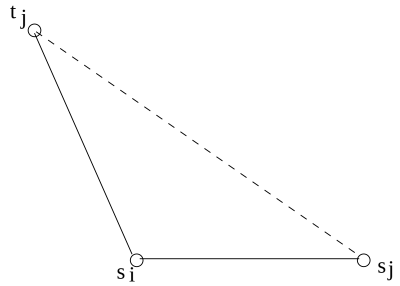
[1]Recall: a semi-metric d has the following properties:

- $d(i,i) = 0$ for all i
- $d(i,j) + d(j,\ell) \geq d(i,\ell)$ for all i, j, ℓ (i.e. triangle inequality)
- $d(i,j) \geq 0$ for all i, j

The difference with a metric is that we can have $d(i,j) = 0$ for $i \neq j$.

i.e., set $S = B(s_i, \rho) = \{j : d(s_i j) \leq \rho\}$, then $(S, \bar{S})$ is a ball cut.

We will repeatedly generate a ball cut around the source of a commodity that is still connected. Now note that the triangle inequality, together with $d(s_j, t_j) \geq 1$ implies



$d(s_i, s_j) + d(s_i, t_j) \geq d(s_j, t_j) \geq 1$, and thus $\max\{d(s_i, s_j), d(s_i, t_j)\} \geq 1/2$. So if we choose the radius ρ to be less than 1/2 when making ball cuts, there will be at most one terminal of each other commodity in the side of the cut with s_i (and hopefully there will be one).

Define

$$\begin{aligned} f_i(\rho) &= \text{capacity of the ball cut of radius } \rho \text{ around } s_i, \text{ and} \\ g_i(\rho) &= \text{volume of } B(i, \rho) \\ &= \int_0^\rho f_i(t)dt + C. \end{aligned}$$

Note that we have assigned a volume of size C to balls with radius 0, and added this constant to all volumes. It turns out that this will be very useful — see a bit later.

Consider the following experiment: For each ball cut we need to create, we'll pick a radius ρ uniformly at random in the interval $(0, 1/2)$. Let's look at $E(f_i(\rho)/g_i(\rho))$ for this algorithm. First of all note that $f_i(\rho) = g_i'(\rho)$, and that the density function of the uniform distribution on $(0, 1/2)$ equals 2 on this interval, so

$$\begin{aligned} E\Big(\frac{f_i(\rho)}{g_i(\rho)}\Big) &= 2\int_0^{1/2} \frac{g_i'(\rho)}{g_i(\rho)} d\rho \\ &= 2\Big[\ln g_i(\rho)\Big]_{\rho=0}^{1/2} \\ &= 2\ln\Big(\frac{g_i(\frac{1}{2})}{g_i(0)}\Big) \\ &\leq 2\ln\Big(\frac{\sum c(ij)d(ij)}{C}\Big). \end{aligned}$$

We want (recall theorem 2)

$$\sum_{ij\in F} c_{ij} \leq \alpha \sum_{ij} c(ij)d(ij),$$

where F is the multicut we are constructing. If we choose $C = (1/k)\sum c(ij)d(ij)$, we get that expected value of the ratio between capacity and volume for each cut is $2\ln k$, and, moreover, assuming we created $m \leq k$ ball cuts,

$$\begin{aligned} C(F) &\leq \sum_{\ell=1}^{m} c(B_\ell, \bar{B}_\ell) \\ &\leq \sum_{\ell=1}^{m} (2\ln k)\mathrm{vol}(B_\ell) \\ &= (2\ln k)\sum_{\ell=1}^{m} \mathrm{vol}(B_\ell) \\ &\leq (2\ln k)\Big(\sum_{ij} c(ij)d(ij) + mC\Big) \\ &\leq (4\ln k)\sum_{ij} c(ij)d(ij) \end{aligned}$$

i.e. we have proved theorem 2.
Remark 1 The bound of $4\ln k$ is tight.
Remark 2 This algorithm does not seem to apply to directed graphs.
Remark 3 The above algorithm is due to Garg, Vazirani and Yannakakis [2].

References

[1] Elias Dahlhaus, David S. Johnson, Christos H. Papadimitriou, P. D. Seymour, and Mihalis Yannakakis. The complexity of multiterminal cuts. *SIAM J. Comput.*, 23(4):864–894, 1994.

[2] Naveen Garg, Vijay V. Vazirani, and Mihalis Yannakakis. Approximate max-flow min-(multi)cut theorems and their applications. *SIAM J. Comput.*, 25(2):235–251, 1996.

[3] Martin Grötschel, László Lovász, and Alexander Schrijver. *Geometric algorithms and combinatorial optimization*, volume 2 of *Algorithms and Combinatorics*. Springer-Verlag, Berlin, second edition, 1993.

[4] L. G. Hačijan. A polynomial algorithm in linear programming. *Dokl. Akad. Nauk SSSR*, 244(5):1093–1096, 1979.