CS 684 Algorithmic Game Theory — **Scribe: Christina Peraki**
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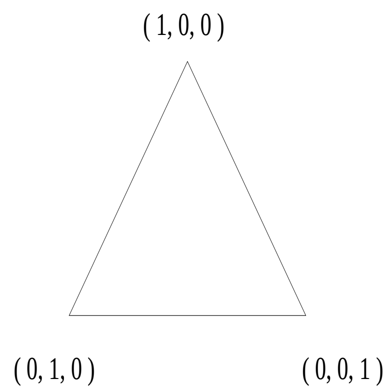
Last time we talked about the Brower fixed point theorem. The goal now is to use this in a game theoretic concept. Let $S_k = \{x = \sum_i \lambda_i e_i, \lambda_i \geq 0, \sum_i \lambda_i = 1\}$, where e_i is the unit vector in coordinate i direction.

Theorem 1 *Let $f : S_k \to S_k$ be a continuous function. Then, there is $x \in S_k$ such that $f(x) = x$.*



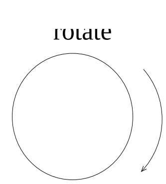
Extension: More generally, (that is, if we want to have some other region), let $f : R \to R$ be a continuous function. Then, there is $x \in R$ such that $f(x) = x$, assuming R is closed, convex and bounded.

This extension is very easy to prove if the convex set can be written in the form of S_k, that is, as a convex combination of a finite set of points. It is also true if instead of a finite set of points, we take an infinite set of points but in a bounded region.

Let us look at the following bad examples when R is either not closed, not convex or not bounded:

1. *not bounded:* Consider a line, which is a closed, convex, but not bounded region of the plane. What is wrong with the line? We can always shift, so there is no fixed point, as is shown in Fig. 1:

2. *not convex:* Consider a circle, which is a closed, bounded, but not convex region of the plane. What is wrong with the circle? We can always rotate, so, again, there is no fixed point, as shown in Fig. 2:



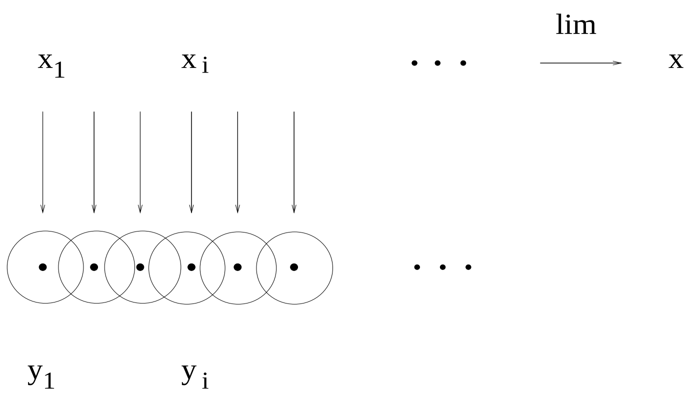
3. *not closed:* Consider an interval, which is a convex, bounded but not closed region. What is wrong with the interval? We cannot have a fixed point, again, by shifting less and less, e.g. $x \to \frac{x}{2} \to ....$

We will use the extension of the fixed point theorem for our game theoretic applications:

Theorem 2 Kakutani fixed point theorem: *Let R be closed, bounded and convex. Let $f : R \to \{$subsets of $R\}$ be a continuous function (for any point x, $f(x)$ is not a point but a region). Then, there is $x \in R$ such that $x \in f(x)$ (now $f(x)$ is a set).*

What does it mean for a set function to be continuous?

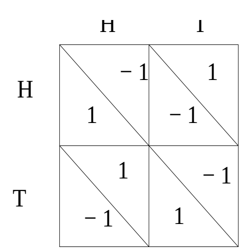
Let $x_1, x_2, ...$ be a sequence with $\lim x_i = x$. If $y_i \in f(x_i)$ (y_i is the best response for x_i), and $\lim y_i$ exists, then $y = \lim y_i \in f(x)$. (weak notion of continuity):



Goal: Does Nash equilibrium exist?

Suppose there are n players and each player i has a fixed set of possible strategies S_i.

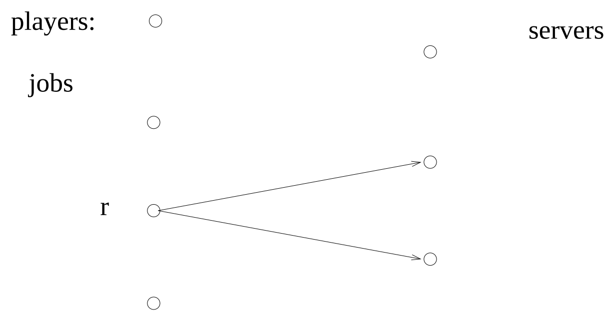
Example: *Matching pennies:* In this case, what is given is the $\text{payoff}_i(s_1, ..., s_n) \in R$ for all



$s_j \in S_j$, and player i. s_1 in this function is a permitted strategy for player 1, s_2 is a permitted strategy for player 2, etc.

Load balancing: The players in this game are the n jobs and each job i is allowed to go to a set $S_i \in S$ of servers, where S is the set of all servers.

Let us consider mixed strategies. A **mixed strategy** is a randomized strategy, that is, a mixed strategy for player j is a probability vector: probabilities p_{js} for all $s \in S_j$ with $p_{js} \geq 0$ and $\sum_{s \in S_j} p_{js} = 1$.



We care about the expected payoff for player i (0 if $p = \frac{1}{2}$ to win and $p = \frac{1}{2}$ to lose):

$$\text{payoff}_i(p_1, ..., p_n) = \text{Expected}(\text{payoff}_i| \text{ strategies chosen } \textbf{independently} \text{ with given probability}),$$

assuming p_j is a probability distribution.

The set of mixed strategies is *Nash* if for all players i and all probabilities p'_i we have:

$$\text{payoff}_i(p_1, ..., p_n) \geq \text{payoff}_i(p_1, ..., p_{i-1}, p'_i, p_{i+1}, ..., p_n),$$

that is, if player i changes to other probabilities, the payoff will not be better.

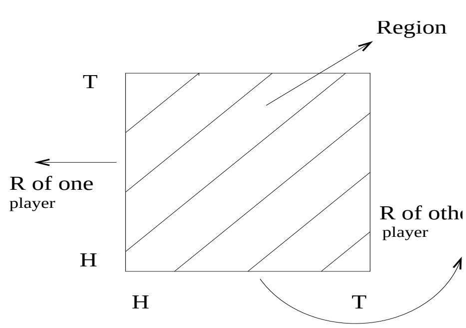
Theorem 3 *(Nash) Nash exists.*

Proof: (The space we are at is precisely the simplex, the probability distribution). Let R be the set of all mixed strategies, that is:

$$R = \{p_{js} \text{ for all } j, \text{ and } s \in S_j, \text{ such that } p_{js} \geq 0, \sum_s p_{js} = 1 \text{ for all } j\}.$$

Goal: Set $f(p_1, ..., p_n) = \{(p'_1, ..., p'_n)$ such that p'_j is the best response for player $j\}$.
Idea: p'_j : player j's best move if other players play $p_1, ..., p_n$.

Example: For matching pennies we have the following figure:



We skatch two proofs. First using Kakutani's fixed point theorem, and then a different proof using only Brouwer's fixed point theorem.
Idea: Apply Kakutani fixed point theorem: There is some $p_1, ..., p_n$ such that $(p_1, ..., p_n) \in f(p_1, ..., p_n)$. (This is exactly what we're looking for! That is the player's best choice).

Is this f continuous? According to the definition given earlier, it is. This finishes the first proof.

Next we want to sketch a proof using only Brouwer's fixed point theorem. For this we need to find p'_{js} for $s \in S_j$ in order to define $f(p_1, ..., p_n) = (p'_1, ..., p'_n)$, so that f is continous, and a fixed point is the equlibrium we are looking for. The idea is to consider,

$$d_{js} = \text{payoff}_j(p_1, ..., p_{i-1}, s, p_{i+1}, ..., p_n) - \text{payoff}_j(p_1, ..., p_n),$$

where $\text{payoff}_j(p_1, ..., p_{i-1}, s, p_{i+1}, ..., p_n)$ is the payoff for switching to the deterministic strategy s.

Let us take the following steps:

- It would seem natural to define p' as follows: $p'_{js} = p_{js} + d_{js}$, where p_{js} is increased if s is better and decreased if worse. However, what if d_{js} is a large negative number?
- To avoid this problem, let $p'_{js} = p_{js} + max(d_{js}, 0)$, where we changed the previous expression because if d_{js} is a large negative number we will no longer get a probability. However, there is still a problem, these "probabilities" do not sum up to one.
- So finally, we define $p'_{js} = \frac{p_{js}+max(d_{js},0)}{\sum_s p_{js}+max(d_{js},0)}$, where we normalized the previous expression to finally get a probability. This is also a continuous function.

In this way, f is continuous, so fixed point exists. Fixed point is Nash.