CS 6815: Lecture 23

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November 15

1 Zig-Zag Product and Explicit Expanders

The Zig-Zag product is a landmark result in pseudorandomness and graph theory by Reimgold, Vadhan & Wigderson that allows for construction of explicit expander graphs.

We have previously have seen some explicit expanders, but they require deep group-theory results about Cayley graphs in order to prove expansion. The Zig-Zag product provides a simple graph-theoretic tool to iteratively construct arbitrarily large expander graphs. The construction has also found recent use in a proof of the PCP theorem.

1.1 Notation

Let H be a (d, D)-graph (i.e. a D-regular graph on n vertices). Then M_H is the adjacency matrix of H and $\hat{M}_H$ is the transition matrix, $\frac{1}{d}M_H$. We denote by $\lambda(H)$ the normalized second eigenvalue, that is the second largest eigenvalue in absolute value of $\hat{M}_H$. The "(normalized) spectral gap" is then $1 - \lambda(H)$. We say H is a (D, d, α)-spectral expander if $\alpha \leq \lambda(H)$.

1.2 Iterative Constructions

For small vertex sets we can brute force over all graphs to find a good expander. This allows us to find seeds for any iterative constructions that we develop.

As a first example of a graph operation which increases expansion, note that for any (D, d, α)-expander H, H^2 is a (D, d^2, α^2) expander, where H^2 is the graph whose adjacency matrix is M_H^2. I.e. it is the graph H but with an edge for every path of length 2 in H. We see that this allows us to increase expansion at the cost of increased degree. We will next see how graph products can be used to decrease degree while preserving expansion.

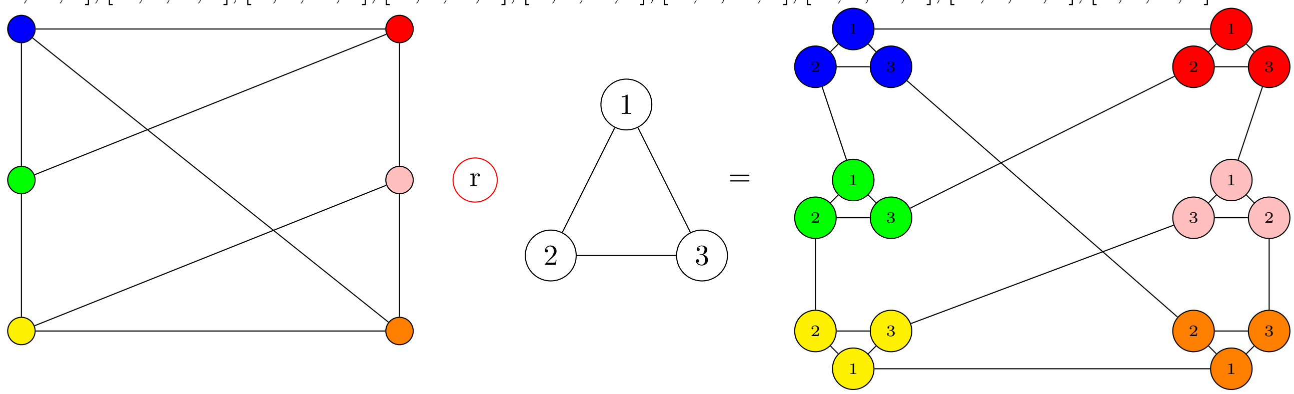
1.3 Replacement Product

Definition 1.1. *Let G be an (n, D)-graph and H be a (D, d)-graph. For every vertex in G, label each edge with a vertex of H. The* ***replacement product*** $G \circledR H$ *has vertex set $G \times H$ and edges $(g_h, g_{h'})$ for all $(h, h') \in H$, as well as $(g_h, g'_{h'})$ whenever $(g, g') \in G$ has label h and (g', g) has label h'. Hence, $G \circledR h$ is an $(nD, d + 1)$-graph*

Conceptually, we replace each vertex in G with a copy of H, and assign each edge out of the vertex to a unique vertex in the cloud. Note that the degree of an iterative family of such graphs is constant if we hold H constant. The replacement product gives an interesting way to produce arbitrarily large constant-degree graphs, but the expansion properties are not suitable.

Below, we give a depiction of a replacement product, where the edges of G are specified in the

following fashion (the format is 4-tuples of $[u, x, v, y]; u, v \in \mathcal{V}(G), x, y \in \mathcal{V}(H)$):
$[B, 2, G, 1]; [G, 2, Y, 2]; [Y, 1, O, 1]; [O, 3, P, 2]; [P, 1, R, 3]; [R, 1, B, 1]; [B, 3, O, 2]; [G, 3, R, 2]; [Y, 3, P, 3]$



1.4 Zig-Zag Product

In order to ensure that the expansion properties of G and H are preserved, we need to make a slight modification to the replacement product. The vertex set of $G \circledz H$ is the same as in the replacement product, but now we have an edge $(g_h, g'_{h'})$ if there is a path $g_h - g_i - g'_j - g'_{h'}$. I.e an edge exists for every path consisting of a small "zig" inside the cloud, then a big "zag" between clouds and finally a small "zig" within the new cloud.

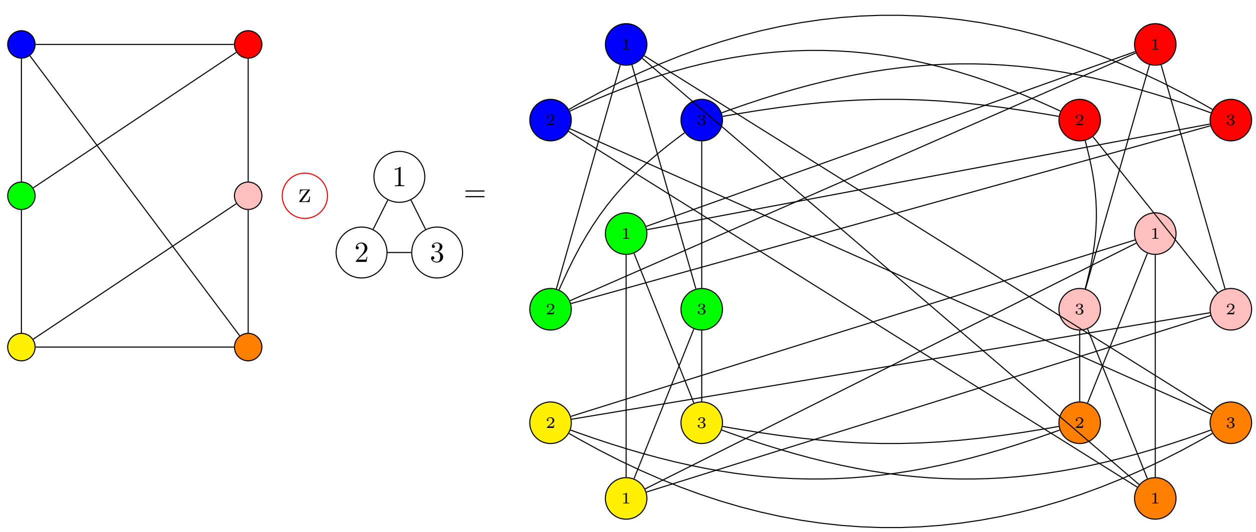
Lemma 1.2. *$G \circledz H$ is a $(ND, d^2, \alpha + \beta + \beta^2)$-graph when G is a (N, D, α) graph and H is a (D, d, β) graph. We remark that the quality of the zig-zag product can be shown to better than what we prove but leverages deeper techniques than what we use. In particular, we can shed the β^2 term.*

Using Lemma 1.2 we want to construct a family of constant-degree expanders. Let H be a $(d^4, d, \frac{1}{8})$-graph and $G_1 = H^2$. Then inductively define $G_{i+1} \coloneqq (G_i)^2 \circledz H$.

Claim 1.3. *G_i is a $(d^{4i}, d^2, \frac{1}{2})$-graph.*

Proof. By induction:
If G_i is a $(d^{4i}, d^2, \frac{1}{2})$-graph, then G_i^2 is a $(d^{4i}, d^4, \frac{1}{4})$-graph and H is a $(d^4, d, \frac{1}{8})$-graph.

Thus, $G_{i+1} = {G_i}^2 \circledz H$ is a $(d^{4i+4}, d^2, \frac{1}{4} + \frac{1}{8} + \frac{1}{64})$-graph, and $\frac{1}{4} + \frac{1}{8} + \frac{1}{64} < \frac{1}{2}$. $\square$



Insight: We want to show that if D is a distribution over the zigzag graph, then the marginal distribution is either near uniform, in which case we want the graph to preserve the near-uniformity, or if D is not near uniform then the entropy of the distribution increases.

Proof. (of 1.2)

Define $\widetilde{M} = \hat{M}_h \otimes I_N$ as the $ND \times ND$ block-diagonal matrix corresponding to the vertex replacement. Then, we want to find a permutation matrix P such that $P \in Mat(ND, ND)$ where $\widetilde{M}P\widetilde{M}$ encodes the zig-zag product and P encodes the zag step. Therefore, $P_{(v,l),(u,m)}$ is non-zero if edge l in cloud v is associated with edge m in cloud u.

To demonstrate the appropriate expansion, we find it is equivalent to show:

$$|x^T \widetilde{M}P\widetilde{M}x| \leq \alpha + \beta + \beta^2 \;\forall x s.t ||x||_2 = 1, x \perp 1_D$$

Here, the second constraint is relevant as 1_D is the all-ones vector and hence parallel to the first eigenvector. As we want to find the second eigenvalue, corresponding to the second eigenvector, it is sufficient to consider such orthogonal choices of x.

Rewriting $x = x^{\parallel} + x^{\perp}$ means the following two observations are useful:

1. ${x^{\parallel}}^T \widetilde{M} = {x^{\parallel}}^T$
2. $||{x^{\perp}}^T \widetilde{M}||_2 \leq \beta ||x^{\perp}||_2$

Additionally, substituting means we are considering three terms(constraints on x are hereafter omitted):

$$|{x^{\parallel}}^T \widetilde{M}P\widetilde{M}x^{\parallel} + 2{x^{\parallel}}^T \widetilde{M}P\widetilde{M}x^{\perp} + {x^{\perp}}^T \widetilde{M}P\widetilde{M}x^{\perp}| \leq \alpha + \beta + \beta^2$$

Working with the terms in the order they appear (the bounds below are subsequently proven):

1. $|{x^{\parallel}}^T \widetilde{M}P\widetilde{M}x^{\parallel}| \leq \alpha$
   Define $y \in \mathbb{R}^n$ w.r.t $x \in \mathbb{R}^{N \times D}$ such that $y(v) = \sqrt{D}x^{\parallel}(v, l)$ (as y averages, the input l is irrelevant).

$$\begin{aligned}
|{x^{\parallel}}^T \widetilde{M}P\widetilde{M}x^{\parallel}| &= |{x^{\parallel}}^T P x^{\parallel}| \\
&= |y^T \hat{M} y| \\
&\leq \alpha ||x^{\parallel}||_2^2 \\
&\leq \alpha
\end{aligned}$$

   A core observation is that taking the norms of y introduces a factor of D (or 2 $\sqrt{D}$ factors that are multiplied but this is cancelled in the $\hat{M}$).

2. $|2{x^{\parallel}}^T \widetilde{M}P\widetilde{M}x^{\perp}| \leq \beta$

$$\begin{aligned}
|2{x^{\parallel}}^T \widetilde{M}P\widetilde{M}x^{\perp}| &= |2{x^{\parallel}}^T P\widetilde{M}x^{\perp}| \\
&\leq ||2{x^{\parallel}}^T P||_2 ||\widetilde{M}x^{\perp}||_2 \\
&\leq 2||{x^{\parallel}}^T||_2 \beta ||x^{\perp}||_2 \\
&\leq 2\beta(\frac{||x^{\parallel}||_2^2 + ||x^{\perp}||_2^2)}{2}) \\
&\leq \beta
\end{aligned}$$

3. $|{x^{\perp}}^T \widetilde{M} P \widetilde{M} x^{\perp}| \leq \beta^2$
   Follows immediately from Cauchy-Schwartz and the 2^{nd} statement regarding $||{x^{\perp}}^T \widetilde{M}||_2$

We apply triangle inequality on the original terms to achieve the final result.

$\square$