CS 6815: Lecture 10

Instructor: Eshan Chattopadhyay Scribe: Lucy Li

September 27, 2018

1 Expander Graphs

More information on expander graphs can be found in Chapter 4 of Salil Vadhan's book [1], and in a survey by Hoory, Linial, and Widgerson [2].

Informally, expander graphs are sparse graphs that are "really well connected."

More formally, an expander graph is a graph $G = (V, E)$, with $|V| = n$, that is:

1. a multigraph.
2. undirected, but each edge counts as 2 edges. $\{u, v\} = (u, v), (v, u)$
3. d-regular.

In addition to the above properties, the graph should be "really well connected." What does it mean to be "really well connected"? Here are some equivalent ideas.

1. $N(S) = \{v \in V : \exists u \in S, (u, v) \in E\}$, and $N(S)$ is large
2. Let $E(S, T)$ be the edges between sets of vertices S and T, and let $\bar{S}$ be $V \setminus S$. $E(S, \bar{S})$ is large.

But what does "large" mean? We will explore the definitions more in the next sections.

2 Edge Expansion

For $S \subseteq V$, let $\partial S = E(S, \bar{S})$, which is the number of edges leaving set S.

Definition 2.1. *The expansion ratio is:*

$$h(G) = \min_{S \subseteq V : |S| \leq \frac{n}{2}} \frac{|\partial S|}{|S|} \tag{1}$$

Definition 2.2. *G is an α edge expander if $h(G) \geq \alpha$.*

3 Vertex Expanders

Definition 3.1. *G is a (K, A) vertex expander if $\forall S \subseteq V, |N(S)| \geq A|S|$, where $|S| \leq K$.*

Theorem 3.2. *For $d \geq 3$, $\exists$ a constant $\alpha > 0$ such that a random d-regular graph is a $(\alpha n, d - \frac{11}{10})$ vertex expander (with high probability).*

Some constructions of vertex expanders:

1. Lubotzky-Phillips-Sarnak [3]
   $V = \mathbb{Z}_p$, where p is a prime.
   $x \in V$ is connected to: $x+1$, $x-1$, x^{-1}. (3-regular graph)
   This construction, while simple, is not ideal because we don't know how to deterministically generate large primes

2. Margulis [4]
   $V = \mathbb{Z}_m \times \mathbb{Z}_m, m \in \mathbb{Z}^+$
   Vertex (x, y) is connected to $(x+y, y), (x-y, y), (x, y+x), (x, y-x), (x+y+1, y), (x, y+x+1), (x, y-x+1)$ (all mod m)

4 Spectral Expansion

Notation: A is the adjacency matrix for graph G. $\hat{A} = \frac{1}{d}A$ is the normalized adjacency matrix.

A is a symmetric real matrix. $Av = \lambda v, v \in \mathbb{R}^n$, where λ is an eigenvalue and v is the corresponding eigenvector.

Fact: Given that $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ are eigenvalues of A, with corresponding eigenvectors $\{v_1, \ldots, v_n\}$:

1. The λ_i's are real.

2. The v_i's form an orthonormal basis.

Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the sorted eigenvalues. Then

1. $\lambda_1 = d, v_1 = \frac{1}{\sqrt{n}}\overrightarrow{\mathbf{1}}$, where $\overrightarrow{\mathbf{1}}$ is the vector of all 1s.

2. $\lambda_1 = \lambda_2$ iff G is not connected.

3. Let $\lambda = \max\{|\lambda_2|, |\lambda_n|\}$.

4. If $\lambda_n = -\lambda_1$, then G is a bipartite graph.

Definition 4.1. *G is a (n, d, t) spectral expander if $\lambda_G \leq t$, where λ_G is the λ value for graph G. The spectral gap is defined to be $d - t$.*

Claim 4.2 (Alon-Boppana)**.** $\lambda \geq 2\sqrt{d-1} - o_n(1)$.

Claim 4.3. *(weaker claim) $\lambda \geq \sqrt{d}(1 - o_n(1))$.*

Proof.

$$\begin{aligned} nd &\leq tr(A^2) \\ &= \sum_i \lambda_i^2 \\ &= d^2 + \sum_{i=2}^{n} \lambda_i^2 \\ &\leq d^2 + \lambda^2(n-1) \\ \lambda &\geq \sqrt{\frac{d(n-d)}{n-1}} \end{aligned}$$

□

Lemma 4.4 (Expander Mixing Lemma)**.** *Let G be a (n, d, λ) spectral expander. Then, $\forall S, T \subseteq V$, $\left|E(S,T) - \frac{d|S||T|}{n}\right| \leq \lambda\sqrt{|S||T|}$.*

Proof. Let $I_S, I_T \in \mathbb{R}^n$, where I_S, I_T are the indicator vectors for S, T respectively. We know that $I_S = \sum \alpha_i \overrightarrow{\mathbf{v}}_i, I_T = \sum \beta_i \overrightarrow{\mathbf{v}}_i$, where $\overrightarrow{\mathbf{v}}_i$ are the eigenvectors of A. Note that $\overrightarrow{\mathbf{v}}_1 = \frac{1}{\sqrt{n}}\overrightarrow{\mathbf{1}}$ and $\langle I_S, \overrightarrow{\mathbf{v}} \rangle = \alpha_1 = \frac{|S|}{\sqrt{n}}$.

$$\begin{aligned}
|E(S,T)| &= I_S^T A I_T \\
&= \sum_{i=1}^{n} \alpha_i \beta_i \lambda_i \\
&= \frac{d|S||T|}{n} + \sum_{i=2}^{n} \alpha_i \beta_i \lambda_i \,.
\end{aligned}$$

This means that

$$\begin{aligned}
\implies \left|E(S,T) - \frac{d|S||T|}{n}\right| &\leq \lambda \sum_{i=2}^{n} \alpha_i \beta_i \\
&\leq \lambda \left(\sum \alpha_i^2\right)^{\frac{1}{2}} \left(\sum \beta_i^2\right)^{\frac{1}{2}} \\
&\leq \lambda |S|^{\frac{1}{2}} |T|^{\frac{1}{2}}.
\end{aligned}$$

□

5 Spectral Expansion $\implies$ Vertex Expansion

Let G be a (n, d, α) spectral expander graph, with $\hat{A}$ as its normalized adjacency matrix, and $\alpha = \frac{\lambda}{d}$.

Let $S \subseteq V$. We want to show that G is a vertex expander by proving that $N(S) \geq A|S|$.

Let P be the probability distribution uniform on S. $P \in \mathbb{R}^n$. $P(i) = \frac{1}{|S|}$ if $i \in S$, and 0 otherwise.

Definition 5.1. *If $p \in \mathbb{R}^n$, the Renyi entropy of p is $H_2(p) = \log\left(\frac{1}{\|p\|_2^2}\right)$.*

The Renyi entropy of P is $H_2(P) = \log(|S|)$.

Claim 5.2. *$|Supp(P)| \geq 2^{H_2(P)}$.*

Proof.

$$\begin{aligned}
1 &= \sum_{i \in \mathrm{Supp}(P)} P(i) \\
&\leq \sqrt{\mathrm{Supp}(P)} \left(\sum P(i)^2\right)^{\frac{1}{2}} \\
&= \sqrt{\mathrm{Supp}(P)} \|P\|_2 \\
\implies \mathrm{Supp}(P) &\geq \frac{1}{\|P\|_2^2} = 2^{H_2(P)}
\end{aligned}$$

□

References

[1] S. P. Vadhan *et al.*, "Pseudorandomness," *Foundations and Trends® in Theoretical Computer Science*, vol. 7, no. 1–3, pp. 1–336, 2012.

[2] S. Hoory, N. Linial, and A. Wigderson, "Expander graphs and their applications," *BULL. AMER. MATH. SOC.*, vol. 43, no. 4, pp. 439–561, 2006.

[3] A. Lubotzky, R. Phillips, and P. Sarnak, "Ramanujan graphs," *Combinatorica*, vol. 8, no. 3, pp. 261–277, 1988.

[4] G. A. Margulis, "Explicit constructions of concentrators," *Problemy Peredachi Informatsii*, vol. 9, no. 4, pp. 71–80, 1973.