CS4860 Applied Logic - 2018-Study Guide

Abstract

This is a companion document to the course Review. It provides further guidance for study and a few sample questions of the kind that will appear on the exam. It mentions the key topics that will be covered on the **final exam** on May 22, from 2pm to 4pm in Rockefeller Hall room 104.

One of the key points to guide your study is that the exam will request a few proofs of propositions in First-Order Logic using **refinement logic** and evidence semantics. It will not ask for tableaux proofs. Classical FOL is distinguished from iFOL by the single rule $A \vee \neg A$ justified by the classical realizer **magic(A)** discussed in Lectures 5 and 6 and used in other lectures and examples, e.g. Lectures 11 and 14.

The magic realizer is used in all of the classical theories such as Peano Arithmetic and classical real analysis, RA, and ZFC. In the case of number theory, we obtain Heyting Arithmetic (HA) from Peano Arithmetic (PA) by not allowing the magic rule.

1 Study Questions

1. A good example to illustrate the difference between classical and constructive logic is the classical tautology $(A \Rightarrow B) \Leftrightarrow (\neg A \vee B)$. One direction of the implication is constructively true and can be proved in refinement logic. Give the proof. The other direction is not constructively valid. Explain why it is not. Using the magic rule, that implication can also be proved. Show this proof.

Does the explanation for this example also work for the following quantified version? Smullyan [1] does not use a symbol for the domain of discourse D. He talks about the universe U of discourse and assumes it is non-empty, see page 46. So he can always reference an element u in U. In constructive logic, this domain D can be empty. That is realistic because we investigate sets such as all odd perfect numbers, say OPN, for which it is not known whether there is even one element.[1]

Here is a constructive formulation of the generalization of the above proposition to iFOL.

$$\forall x : D.((A(x) \Rightarrow B(x)) \Leftrightarrow (\neg A(x) \vee B(x))).$$

2. Consider this proposition: $(\neg \forall x : D.P(x)) \Rightarrow \exists x : D.\neg P(x)$. Is this true in FOL? Is this true in iFOL? Explain.

3. Consider Smullyan's proof on page 55 of

$$\forall x : D(P(x) \Rightarrow Q(x)) \Rightarrow (\forall x : D.P(x) \Rightarrow \forall x : D.Q(x)).$$

Give a refinement logic proof using magic as necessary.

4. Prove this exercise on Smullyan page 56 using refinement logic.

$$\exists y : D.(P(y) \Rightarrow \forall x : D.P(x)).$$

Explain the result intuitively. At first glance it seems unprovable and untrue.

5.Is the following proposition constructively valid? Is it valid using magic, e.g. classically valid?

$$(\exists x : D.P(x) \& \exists x : D.Q(x)) \Rightarrow \exists x : D.(P(x) \& Q(x)).$$

6. State König's Lemma and sketch a classical proof. For extra credit comment on its constructive status.

[1]This question has been open since Euclid. It has been checked for numbers up to 10^{300} at least, and none have been found.

References

[1] R. M. Smullyan. *First–Order Logic*. Springer-Verlag, New York, 1968.