We say that the substitution of t for x in A(x) is *free*, when t is free for x in A(x). With only the smattering of interpretation indicated above, it should be clear that a substitution is inappropriate when it is not free.

The two formulas in (I) mean the same; but the two in (II) do not.

For an informal example, consider the second expression of (A) or (C). This stands for a function of $y$, call it

(G) $$f(y) = \lim_{x\to 0} f(x, y) = \lim_{z\to 0} f(z, y).$$

The value of $f(y)$ for $y = z$ is then given properly by

(H) $$f(z) = \lim_{x\to 0} f(x, z), \quad \text{not by} \quad f(z) = \lim_{z\to 0} f(z, z).$$

Example 9. To illustrate the handling of the terminology and notations explained in this section, say that x is (i.e. "x" denotes) a variable, A(x) is (i.e. "A(x)" denotes) a formula, and b is (i.e. "b" denotes) a variable such that (i) b is free for x in A(x) and (ii) b does not occur free in A(x) (unless b is x). Under our substitution notation, since "x" and "A(x)" are introduced first, (iii) A(b) is (by definition) the result of substituting b for (the free occurrences of) x in A(x). By (i), the occurrences of b in A(b) which are introduced by this substitution are free. By (ii), there are no other free occurrences of b in A(b). Thus the free occurrences of b in A(b) are exactly the occurrences introduced by the substitution. Hence (inversely to (i) — (iii)): (iv) x is free for b in A(b), (v) x does not occur free in A(b) (unless x is b), and (vi) A(x) is (in fact) the result of substituting x for (the free occurrences of) b in A(b). To make this example particular,

x, A(x), b, A(b)

may be respectively,

$$c, \quad \exists c(c'+a=b) \supset \neg a=b+c, \quad d, \quad \exists c(c'+a=b) \supset \neg a=b+d.$$

**§ 19. Transformation rules.** In this section we shall introduce further metamathematical definitions (called *deductive rules* or *transformation rules*) which give the formal system the structure of a deductive theory. To emphasize the analogy to an informal deductive theory, we shall start with a list of 'postulates'; however, for the metamathematics, these are not postulates in the sense of assumptions, as indeed they cannot be when officially they have no meaning, but only formulas and forms (or schemata) to which we shall refer when we give the definitions.



Before giving the postulate list, let us illustrate the types of postulates which will appear in the list. The simplest is an 'axiom', of which $\neg a'=0$ is an example. This is a formula of the formal system. Then we may have an 'axiom form' or 'axiom schema', of which "B ⊃ A ∨ B" is an example. This is a metamathematical expression, which gives a particular axiom each time formulas are specified as represented by the metamathematical letters "A" and "B". For example, when A is $a'=0$ and B is $\neg a'=0$, we obtain the axiom $\neg a'=0 \supset a'=0 \vee \neg a'=0$. The axiom schema is thus a metamathematical device for specifying an infinite class of axioms having a common form.

We must also have another kind of postulates, which formalize the operations of deducing further theorems from the axioms. These are the 'rules of inference', of which the following is an example:

$$\frac{A, \quad A \supset B}{B.}$$

This is a schema containing three metamathematical expressions "A", "A ⊃ B" and "B", which represent formulas whenever formulas are specified as represented by the metamathematical letters "A" and "B". The sense of the rule is that the formula represented by the expression written below the line may be 'inferred' from the pair of formulas represented by the two expressions written above the line. For example, by taking as A the formula $\neg a'=0$ and as B the formula $a'=0 \vee \neg a'=0$, the rule allows the inference from $\neg a'=0$ and $\neg a'=0 \supset a'=0 \vee \neg a'=0$ to $a'=0 \vee \neg a'=0$. Since $\neg a'=0$ and $\neg a'=0 \supset a'=0 \vee \neg a'=0$ are axioms (as we just saw), $a'=0 \vee \neg a'=0$ is a further 'formal theorem'. (Our terminology will include the axioms as theorems.)

We shall now display the full postulate list, and then give the definitions establishing the deductive structure of the formal system by referring to the list. The reader may verify that the cumulative effect of the series of definitions will be to define a subclass of the class of formulas called 'provable formulas' or 'formal theorems'.

### Postulates for the Formal System

Dramatis personae. For Postulates 1—8, A, B and C are formulas. For Postulates 9—13, x is a variable, A(x) is a formula, C is a formula which does not contain x free, and t is a term which is free for x in A(x).

GROUP A. Postulates for the predicate calculus.

GROUP A1. Postulates for the propositional calculus.

1a. $A \supset (B \supset A)$.

1b. $(A \supset B) \supset ((A \supset (B \supset C)) \supset (A \supset C))$.

2. $\dfrac{A,\ A \supset B}{B.}$

3. $A \supset (B \supset A \,\&\, B)$.

4a. $A \,\&\, B \supset A$.

4b. $A \,\&\, B \supset B$.

5a. $A \supset A \vee B$.

5b. $B \supset A \vee B$.

6. $(A \supset C) \supset ((B \supset C) \supset (A \vee B \supset C))$.

7. $(A \supset B) \supset ((A \supset \neg B) \supset \neg A)$.

8°. $\neg\neg A \supset A$.

GROUP A2. (Additional) Postulates for the predicate calculus.

9. $\dfrac{C \supset A(x)}{C \supset \forall x A(x).}$

10. $\forall x A(x) \supset A(t)$.

11. $A(t) \supset \exists x A(x)$.

12. $\dfrac{A(x) \supset C}{\exists x A(x) \supset C.}$

GROUP B. (Additional) Postulates for number theory.

13. $A(0) \,\&\, \forall x(A(x) \supset A(x')) \supset A(x)$.

14. $a'=b' \supset a=b$.

15. $\neg a'=0$.

16. $a=b \supset (a=c \supset b=c)$.

17. $a=b \supset a'=b'$.

18. $a+0=a$.

19. $a+b'=(a+b)'$.

20. $a \cdot 0=0$.

21. $a \cdot b'=a \cdot b+a$.

(The reason for writing "°" on Postulate 8 will be given in § 23.)

One may verify that 14—21 are formulas; and that 1—13 (or in the case of 2, 9 and 12, the expression(s) above, and the expression below, the line) are formulas, for each choice of the A, B, C, or x, A(x), C, t, subject to the stipulations given at the head of the postulate list.

The class of 'axioms' is defined thus. A formula is an *axiom*, if it has one of the forms 1a, 1b, 3—8, 10, 11, 13 or if it is one of the formulas 14—21.

The relation of 'immediate consequence' is defined thus. A formula is an *immediate consequence* of one or two other formulas, if it has the form shown below the line, while the other(s) have the form(s) shown above the line, in 2, 9 or 12.

This is the basic metamathematical definition corresponding to Postulates 2, 9 and 12, but we shall restate it with additional terminology



which draws attention to the process of applying the definition. Postulates 2, 9 and 12 we call the *rules of inference*. For any (fixed) choice of the A and B, or the x, A(x) and C, subject to the stipulations, the formula(s) shown above the line is the *premise* (are the *first* and *second premise*, respectively), and the formula shown below the line is the *conclusion*, for the *application* of the rule (or the (*formal*) *inference* by the rule). The conclusion is an *immediate consequence* of the premise(s) (by the rule).

Carnap 1934 brings the two kinds of postulates under the common term 'transformation rules', by considering the axioms as the result of transformation from zero premises.

The definition of a '(formally) provable formula' or '(formal) theorem' can now be given inductively as follows.

1. If D is an axiom, then D is *provable*. 2. If E is *provable*, and D is an immediate consequence of E, then D is *provable*. 3. If E and F are *provable*, and D is an immediate consequence of E and F, then D is *provable*. 4. A formula is *provable* only as required by 1—3.

The notion can also be reached by using the intermediate concept of a '(formal) proof', thus. A (*formal*) *proof* is a finite sequence of one or more (occurrences of) formulas such that each formula of the sequence is either an axiom or an immediate consequence of preceding formulas of the sequence. A proof is said to be a proof *of* its last formula, and this formula is said to be (*formally*) *provable* or to be a (*formal*) *theorem*.

EXAMPLE 1. The following sequence of 17 formulas is a proof of the formula $a=a$. Formula 1 is Axiom 16. Formula 2 is an axiom, by an application of Axiom Schema 1a in which the A and the B of the schema are both $0=0$; and Formula 3 by an application in which the A is $a=b \supset (a=c \supset b=c)$ and the B is $0=0 \supset (0=0 \supset 0=0)$. Formula 4 is an immediate consequence of Formulas 1 and 3, as first and second premise respectively, by an application of Rule 2 in which the A of the rule is $a=b \supset (a=b \supset b=c)$ and the B is $[0=0 \supset (0=0 \supset 0=0)] \supset [a=b \supset (a=c \supset b=c)]$. Formula 5 is an immediate consequence of Formula 4, by an application of Rule 9 in which the x is $c$, the A(x) is $a=b \supset (a=c \supset b=c)$, and the C is $0=0 \supset (0=0 \supset 0=0)$ (which, note, does not contain the x free). Formula 9 is an axiom by an application of Axiom Schema 10, in which the x is $a$, the A(x) is $\forall b \forall c[a=b \supset (a=c \supset b=c)]$, and the t is $a+0$ (which, note, is free for the x in the A(x)). The A(t), by our substitution notation (§ 18), is the result of substituting the t for (the free occurrences of) the x in the A(x), i.e. here the A(t) is $\forall b \forall c[a+0=b \supset (a+0=c \supset b=c)]$.