Evidence Semantics and Elementary Geometry

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1 Introduction

Euclid's methodology of geometry, as presented in the *Elements*, is expressed faithfully using a computational semantics for intuitionistic logic. We consider a faithful expression of Euclid's methodology to be one in which proofs and constructions have an embedded correspondence.

2 Euclid's Propositions

Consider a few of Euclid's propositions:

- Proposition 1 : To construct an equilateral triangle on a finite straight line.
- Proposition 2 : To place a straight line equal to a given straight line with one end at a given point.
- Proposition 4 : If two triangles have two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, then they also have the base equal to the base, the triangle equals the triangle, and the remaining angles equal the remaining angles respectively, namely those opposite the equal sides.
- Proposition 6 : If in a triangle two angles equal one another, then the sides opposite the equal angles also equal one another.
- Proposition 9: To bisect a given rectilinear angle.

We can explain the meaning of these propositions by describing the evidence that can be given for them. The evidence for Propositions 1, 2, and 9, is an explicit construction of a geometric object. The evidence for Propositions 4 and 6 is the demonstration that certain relations hold for a given object. In literature on the *Elements*, it is common to refer to the former type of propositions as *problems* and the latter as *theorems*.

3 Postulates

The evidence for a problem is the construction of a geometric object and the verification that the geometric object has the properties specified in the statement of the problem. The geometric object is constructed by applications of the straightedge and compass. In the *Elements*, the admissible applications of the straightedge and compass are given as postulates. There are five, but we will list only the first three here:

1. To draw a straight line form any point to any point
2. To produce a finite straight line continuously in a straight line
3. To describe a circle with any center and distance

These postulates can be given a functional reading:

1. Postulate 1 specifies a function with two points as inputs and a line as output
2. Postulate 2 specifies a function with a straight line as input and the extension of the straight line as output.
3. Postulate 3 specifies a function with a point and finite straight line (distance) as input and a circle as output.

Thus, we expect that the evidence for a problem from the *Elements* will consist, at least in part, of functions corresponding to the postulates described above.

We will focus on propositions of the form of problems here, because their form coincides with programming problems. Specifically, both geometric construction problems and programming problems require the construction of a (computable) function that transforms an object of the input type into an object of the output type that satisfies some property [1].

4 Evidence Semantics

Evidence semantics provides a computational explanation of the meaning of propositions in terms of the *evidence* that can be given for them.

How can we formally express propositions and their evidence? Consider the propositional variable A which represents some arbitrary proposition. Then we will denote specific evidence for A by a. We can then abbreviate the phrase "a is evidence for A" by $a : [A]$ (or $a \in [A]$), where $[A]$ is the type of evidences for A [2].

Proposition	Evidence Type	Evidence Term
$A \Rightarrow B$	function space: $A \rightarrow B$	$\lambda a.b$
$A \ \wedge \ B$	Cartesian product: $A \times B$	(a, b)
$A \ \vee \ B$	disjoint union: $A + B$	$inl(a), inr(b)$
$\forall x : A.B(x)$	dependent function space: $x : A \rightarrow B(x)$	$\lambda a.b$
$\exists x : A.B(x)$	dependent product: $x : A \times B(x)$	(a, b)
$\perp$	empty set : $\{\}$	0

4.1 Application: Euclid's Propositions

In general, problems from the *Elements* are of the form $\forall x : A \ \exists y : B(x).C(x, y)$ and theorems are of the form $\forall x : A.B(x)$. This is illustrated by the following example.

Consider Proposition 1 from Book I of the *Elements*:

Example 4.1. Proposition 1 : To construct an equilateral triangle on a finite straight line.

The evidence for this proposition is a function taking a straight line into a *pair* consisting of a geometric object and a proof that the geometric object is an equilateral triangle. The function will correspond to applications of the straightedge and compass postulates.

5 Formalizing the Straightedge and Compass

5.1 The Predicates

We take only one type of primitive geometric object: points. On points we take as primitive the binary *apartness* relation and the ternary *leftness* relation. *Leftness* establishes the notion of *orientation* on our Euclidean Plane.

Definition 1. A *Euclidean Plane* structure has a primitive type Point together with the following relations for any $a, b, c, d \in$ Point.

Congruence, written $ab \cong cd$, says that segments ab and cd have the same length.

Betweenness, written a_b_c, says that the point b lies between a and c. This relation is not *strict*, so b could be equivalent to either a or c.

Apartness is a binary relation, signified by #, on points. If $a\#b$ we say that a is *separated* from b.

Leftness is a ternary relation on points, written a left of bc, and says that the point a is to the *left* of the line bc (by bc we mean the *directed* line from b to c).

5.2 The Postulates

The *Magnifying glass*: $M(a,b,c)$ is the function that, by *magnification*, decides whether $c\#a$ or $c\#b$. Thus $M(a,b,c) \in c\#a + c\#b$.

***Orientation*:** requires that a point that is separated from a line is either to the *left or right* of the line. Thus, for any $a, b, c \in$ Point, the function LeftOrRight(a, b, c) takes evidence for $a\#bc$ into evidence for orientation: LeftOrRight$(a, b, c) \in (a$ left of $bc + a$ right of $bc)$.

Straightedge-Straightedge"SS" : formalizes the construction by straightedge of the point of intersection of two lines. For any
$a, b, x,$ *and* $y \in$ Point, if x left of ab and y left of ba a straightedge can be applied twice, once to construct the line ab and again to construct the line xy, in order to determine the point of intersection of ab and xy. We let $SS(a, b, x, y)$ be the point z constructed by the SS axiom.

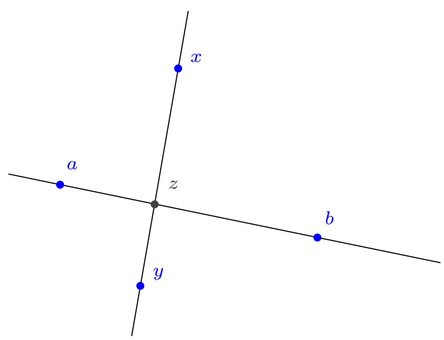

Figure 1: The Straightedge-Straightedge constructor for $z = SS(a, b, x, y)$.

Straightedge Compass "SC": formalizes the extension of a segment by application of a collapsing compass. For any $a, b, c,$ *and* $d \in$ Point, SC constructs from line ab and circle $C(b, d)$ with radius d centered on b, points u and v, where u is on the opposite side of a from b and v is on the same side of b as a. The functions SCO(a, b, c, d) and SCS(a, b, c, d) (where "O" abbreviates opposite and "S" abbreviates same) construct these two points.

Compass-Compass "CC" : formalizes the compass construction of two points resulting from the intersection of two distinct, strictly overlapping circles. For any $a, b, c,$ *and* $d \in$ Point the CC axiom determines the circles $C(a, b)$ and $C(c, d)$ ($C(a, b)$ of radius b centered on a and $C(c, d)$ of radius d centered on c) and constructs u *and* $v \in$ Point where u left of ac and v left of ca such that u and v lie on both $C(a, b)$ and $C(c, d)$. The functions CCL(a, b, c, d) and CCR(a, b, c, d) construct u and v, respectively. The CC construction is demonstrated in figure 3.

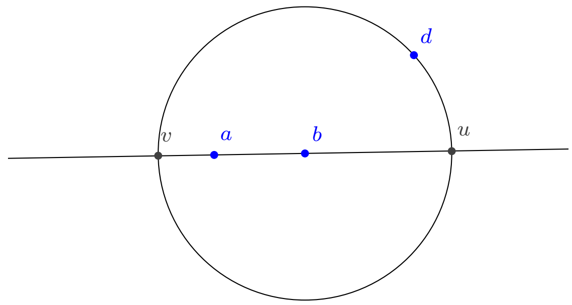

Figure 2: The Straightedge Compass constructor: $u = \mathrm{SCO}(a, b, c, d)$ and $v = \mathrm{SCS}(a, b, c, d)$

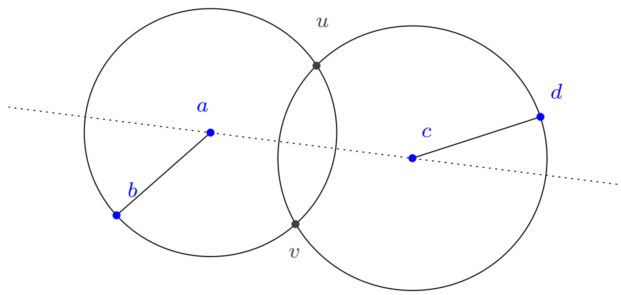

Figure 3: The Compass-Compass constructor for the points u and v such that u left of ac and v right of ac: $u = \mathrm{CCL}(a, b, c, d)$ *and* $v = \mathrm{CCR}(a, b, c, d)$.

***Non-triviality* :** guarantees that there exist two separated points, O and X such that $O\#X$.

5.3 Nuprl Extracts

We begin with Euclid's first proposition, which constructs an equilateral triangle:

> *To construct an equilateral triangle on a given finite straight line.*

Our formal statement of Proposition 1 supposes that $a\#b$ is the given (non-degenerate) "finite straight line." We include an extra assertion requiring the construction of a non-degenerate equilateral triangle, with an apex that lies *to the left* of ab.

Proposition 1.

$$\forall a\colon\text{Point}.\ \forall b\colon\{\text{Point} \mid b\#a\}.\ \exists c\colon\{\text{Point} \mid \text{Cong3}(a,b,c) \wedge c \text{ left of } ab\}$$

where

$$\text{Cong3}(a,b,c) = ab \cong bc\ \wedge\ bc \cong ca\ \wedge\ ca \cong ab.$$

We easily prove Proposition 1 as Euclid does by using the Compass-Compass (CC) Postulate with circles $Circle(a,b)$ and $Circle(b,a)$. This constructs two equilateral triangles, and we take the one where c left of ab.

Our Nuprl extract reflects the simplicity of the construction:

$$\lambda a.\lambda b.\ \text{CCL}(a,b,b,a).$$

So we can define

$$\triangle(a,b) = CCL(a,b,b,a)$$

as the program for Euclid's proposition 1, where CCL is the *Compass-Compass left* constructor from Section **??**. See Figure 4.

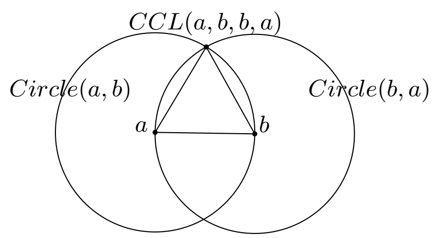

Figure 4: Proposition 1.

5.4 Concluding Remarks

We have introduced evidence semantics as a way to express geometric constructions problems from the *Elements*. Both geometric construction problems and programming problems require the construction of a (computable) function that transforms an object of the input type into an object of the output type that satisfies some property [1]. Thus, the semantics introduced here are generally suitable for expressing constructions.

References

[1] P. Mäenpää and J. von Plato, "The logic of Euclidean construction procedures," *Acta Philos. Fenn*, vol. 49, pp. 275–293, 1990.

[2] R. L. Constable, "The Semantics of Evidence," Cornell University, Ithaca, NY, Tech. Rep., 1985.