9.2 Useful formulas

summations

$$\sum_{i=0}^{\infty} a^i = 1 + a + a^2 + \cdots = \frac{1}{1-a} \quad |a| < 1$$

$$\sum_{i=0}^{\infty} i a^i = a + 2a^2 + 3a^3 \cdots = \frac{a}{(1-a)^2} \quad |a| < 1$$

$$\sum_{i=0}^{\infty} i^2 a^i = a + 4a^2 + 9a^3 \cdots = \frac{a(1+a)}{(1-a)^3} \quad |a| < 1$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

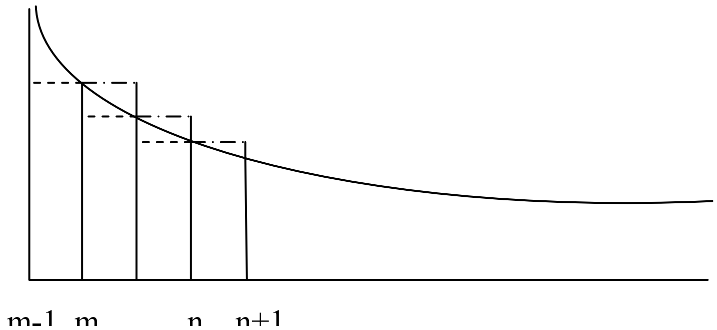
$$\sum_{i=1}^{\infty} \frac{1}{i} = 1 + \tfrac{1}{2} + \left(\tfrac{1}{3} + \tfrac{1}{4}\right) + \left(\tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8}\right) + \cdots \ge 1 + \tfrac{1}{2} + \tfrac{1}{2} + \cdots \text{ and thus diverges}$$

The summation $\sum_{i=1}^{n} \frac{1}{i}$ grows as $\ln n$. $\lim_{i\to\infty}\left(\sum_{i=1}^{n} \frac{1}{i} - \ln(n)\right) = \gamma$ where $\gamma \cong 0.5772$ is Euler's constant. Thus $\sum_{i=1}^{n} \frac{1}{i} \cong \ln(n) + \gamma$ for large n.

$$\sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6}$$

For monotonic decreasing f(x) $\int_{x=m}^{n+1} f(x)dx \le \sum_{i=m}^{n} f(i) \le \int_{x=m-1}^{n} f(x)dx$. Thus

$$\int_{x=2}^{n+1} \tfrac{1}{x^2}\,dx \le \sum_{i=2}^{n} \tfrac{1}{i^2} = \tfrac{1}{4} + \tfrac{1}{9} + \cdots + \tfrac{1}{n^2} \le \int_{x=1}^{n} \tfrac{1}{x^2}\,dx \text{ and hence } \tfrac{3}{2} - \tfrac{1}{n+1} \le \sum_{i=1}^{n} \tfrac{1}{i^2} \le 2 - \tfrac{1}{n}.$$



exponentials and logs

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ e=2.7182 $\frac{1}{e} = 0.3679$

Setting x=1 in $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ yields $\sum_{i=0}^{\infty} \frac{1}{i!} = e$.

$$\lim_{n\to\infty} \left(1 + \tfrac{a}{n}\right)^n = e^a$$

$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 \cdots$ $|x| < 1$

The above expression with –x substituted for x gives rise to the approximations ln(1-x)<-x and $\ln(1-x) > -x - x^2$ 0<x<0.69. The function $f = \ln(1-x) + x + x^2$ goes from 0 to minus infinite as x goes from 1 to 0. It thus crosses zero at least once. The derivative, $\frac{-1}{1-x} + 1 + x = \frac{-x^2}{1-x}$ goes from minus infinity to 0 as x goes from 1 to 0. Thus f has at most one zero in the region and it is for x>0.69.

$$\left(1+x\right)\ln(1+x) = x + \left(1 - \frac{1}{2}\right)x^2 + \left(\frac{1}{2} - \frac{1}{3}\right)x^3 \left(\frac{1}{3} - \frac{1}{4}\right)x^4 \cdots$$

Thus $\left(1+x\right)^{1+x} \le e^{x + \frac{x^2}{2}}$.

Miscellaneous

$$\prod_{i=1}^{n-1}\left(1-\frac{1}{i+1}\right)=\prod_{i=1}^{n-1}\frac{i}{i+1}=\frac{1}{2}\frac{2}{3}\cdots\frac{n-1}{n}=\frac{1}{n}$$

$(1-x)^{1-x} > e^{-x+\frac{x^2}{2}}$ used in Karp

$\frac{n(n-1)\cdots(n-k)}{n^k}=O(1)e^{-\frac{k^2}{2n}}$ See Palmer p129-130.

Proof that $\frac{e^{\delta}}{(1+\delta)^{1+\delta}}<1$ for $\delta>0$. Let $f(\delta)=\ln\frac{e^{\delta}}{(1+\delta)^{1+\delta}}=\delta-(1+\delta)\ln(1+\delta)$. Now $f'(\delta)=-\ln(1+\delta)$ is negative for $\delta>0$. Thus $f(\delta)$ is monotonically decreasing and f(0)=0. Thus $f(\delta)<0$ for $\delta>0$. Hence $\frac{e^{\delta}}{(1+\delta)^{1+\delta}}<1$ for $\delta>0$.

Exercise: What is $\lim_{k\to\infty}\left(\frac{k-1}{k-2}\right)^{k-2}$.

Answer: $\left(1-\frac{1}{k-2}\right)^{k-2}=e$.

Exercise: $e^{-\frac{x^2}{2}}$ has value 1 at x=0 and drops off very fast as x increases. Suppose we wished to approximate $e^{-\frac{x^2}{2}}$ by a function f(x) where

$$f(x)=\begin{cases}1 & |x|\le a\\ 0 & |x|>a\end{cases}.$$

What value of a should we use? What is the integral of the error between f(x) and $e^{-\frac{x^2}{2}}$?

Solution: $\int_{x=-\infty}^{\infty}e^{-\frac{x^2}{2}}dx=\sqrt{2\pi}$. Thus if we select $a=\frac{1}{2}\sqrt{2\pi}$ or approximately 1.25 we will have $\int_{x=-\infty}^{\infty}f(x)\,dx=\sqrt{2\pi}$. The error will be $4\int_{x=-\infty}^{a}e^{-\frac{x^2}{2}}dx=4(0.1056)=0.42$ out of an area of $\sqrt{2\pi}\cong 2.5$ or 17%. ■

trigonometric identities

$$\sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y)$$

$$\cos(x \pm y) = cos(x)\cos(y) \mp \sin(x)\sin(y)$$

$$\cos\left(2\theta\right) = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta$$
$$\sin\left(2\theta\right) = 2\sin\theta\cos\theta$$
$$\sin^2\tfrac{\theta}{2} = \tfrac{1}{2}\left(1 - \cos\theta\right)$$
$$\cos^2\tfrac{\theta}{2} = \tfrac{1}{2}\left(1 + \cos\theta\right)$$

integrals

$$\int \tfrac{1}{a^2+x^2}\,dx = \tfrac{1}{a}\tan^{-1}\tfrac{x}{a} \quad \text{thus} \quad \int_{-\infty}^{\infty} \tfrac{1}{a^2+x^2}\,dx = \tfrac{\pi}{a}\,.$$

$$\int_{-\infty}^{\infty} e^{-\frac{a^2x^2}{2}}\,dx = \frac{\sqrt{2\pi}}{a} \qquad \text{thus} \quad \frac{a}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{a^2x^2}{2}}\,dx = 1$$

$$\int_0^{\infty} x^2 e^{-ax^2}\,dx = \frac{\sqrt{\pi}}{4a\sqrt{a}} = \frac{1}{4a}\sqrt{\frac{\pi}{a}}$$

$$\int_0^{\infty} x^{2n} e^{-ax^2}\,dx = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^{n+1}a^n}\sqrt{\frac{\pi}{a}}$$

$$\int \sin^n\theta\,d\theta = -\frac{sin^{n-1}\theta\cos\theta}{n} + \frac{n-1}{n}\int \sin^{n-2}\theta\,d\theta \quad \text{thus} \quad \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^n\theta\,d\theta = \tfrac{n-1}{n}\tfrac{n-3}{n-2}\cdots$$

To verify take derivatives with respect to θ.

$$\sin^n\theta = \frac{\sin^n\theta}{n} - \frac{n-1}{n}\sin^{n-2}\theta\cos^2\theta + \frac{n-1}{n}\sin^{n-2}\theta$$

$$\sin^n\theta = \frac{\sin^n}{n} + \frac{n-1}{n}\sin^{n-2}\theta\left(1 - \cos^2\theta\right)$$

$$\sin^n\theta = \sin^n\theta$$

binomial coefficients

$$\binom{n}{d}+\binom{n}{d+1}=\binom{n+1}{d+1}$$

The number of ways of choosing k items from 2n equals the number of ways of choosing i items from the first n and choosing k-i items from the second n summed over all i, $0 \le i \le k$.

$$\sum_{i-0}^{k}\binom{n}{i}\binom{n}{k-i}=\binom{2n}{k}$$

Alternatively equate the coefficient of x^k in $(1+x)^n(1+x)^n=(1+x)^{2n}$.

Setting k=n

$$\sum_{i=1}^{n}\binom{n}{i}^2=\binom{2n}{n}$$

More generally $\sum_{i=0}^{k}\binom{n_1}{i}\binom{n_2}{k-i}=\binom{n_1+n_2}{k}$ for the same reason as above.

Stirling approximation

$$n!\cong\left(\frac{n}{e}\right)^n\sqrt{2\pi n} \qquad\qquad \binom{2n}{n}\cong\frac{1}{\sqrt{\pi n}}2^{2n}$$

$$\ln(n!)\cong n\ln n-n$$

$$\sqrt{2\pi n}\frac{n^n}{e^n}<n!<\sqrt{2\pi n}\frac{n^n}{e^n}\left(1+\frac{1}{12n-1}\right)$$

inserted material

$\binom{n}{k}\left(\frac{1}{2}\right)^k\left(\frac{1}{2}\right)^{n-k} \cong \frac{1}{\sqrt{\pi n/2}} e^{-\frac{\left(\frac{n}{2}-k\right)^2}{n/2}}$ is an excellent approximation. Develop how approximation was derived. Needed in Sec 1.1 Chapter 1 see also Central Limit Theorem

inequalities

triangle inequality

$$|x_1 + x_2| \le |x_1| + |x_2|$$

Cauchy-Schwartz inequality

$$\left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right) \ge \left(\sum_{i=1}^{n} x_i y_i\right)^2$$

In vector form $|x||y| \ge |x||y|\cos\theta = x^T y$

Chebyshev sum inequality

If $x_i, y_i \ge 0,\ 1 \le i \le n$ then

$$n\sum_{i=1}^{n} x_i y_i \ge \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)$$

Setting $x_i = y_i$ gives the form

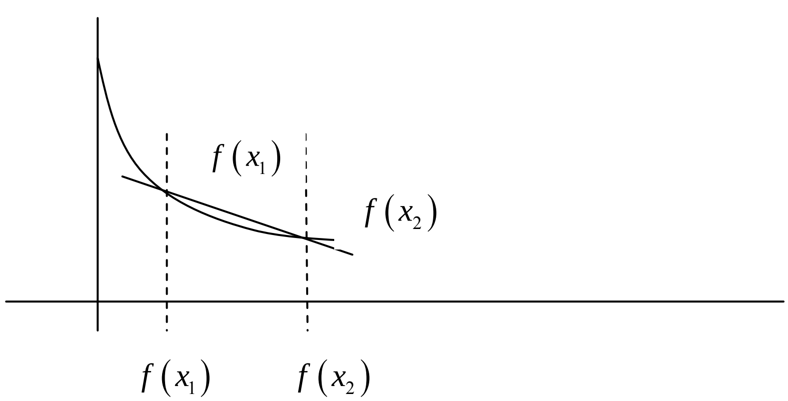
$$n\sum_{i=1}^{n} x_i^2 \ge \left(\sum_{i=1}^{n} x_i\right)^2$$

The above formula can be derived by generalizing the following technique.

$(x_1 - x_2)^2 \ge 0$. Thus $x_1^2 + x_2^2 \ge 2x_1x_2$. Hence $(x_1 + x_2)^2 = x_1^2 + 2x_1x_2 + x_2^2 \le 2x_1^2 + 2x_2^2$.

Jensen's inequality

For convex function f $f(x_1) + f(x_2) \ge 2f\left(\frac{x_1+x_2}{2}\right)$.



More generally for any convex function f, $\sum \alpha_i f(x_i) \geq f\left(\sum \alpha_i x_i\right)$ where $0 \leq \alpha_i \leq 1$ and $\sum_{i=1}^{n} \alpha_i = 1$. It follows that $E\left(f(x)\right) \geq f\left(E(x)\right)$.

Example: Let $f(x) = x^k$. Then $(x_1^k + x_2^k + \cdots + x_n^k) \leq (x_1 + x_2 + \cdots + x_n)^k$ for $x_i \geq 0$ and $E(x) \leq \sqrt[k]{E\left(x^k\right)}$.

Example: Since $f(x) = x^2$ is convex $\left(x_1 + x_2\right)^2 \leq x_1^2 + x_2^2$. When $\lambda_i = \frac{1}{n}$,
$f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \leq \frac{1}{n}\sum_{i=1}^{n} f(x_i)$. Jensen's inequality is derived from this and says that $E\left(f(x)\right) \geq f\left(E(x)\right)$.