Colorimetry as Linear Algebra

CS 465 Lecture 23

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Murdoch redux

- RGB colors add as vectors
  - so do primary spectra in additive display (CRT, LCD, etc.)
- Chromaticity: color ratios (r = R/(R+G+B), etc.)
  - color without regard for overall brightness
- Color matching and metamers
  - any spectrum can be matched by combining 3 primaries
  - metamers: different spectra that look the same
- CIE colorimetry
  - X, Y, and Z: standardized hypothetical primaries
  - $\bar{x}$, $\bar{y}$, $\bar{z}$: color matching functions for X, Y, Z

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Approaching color mathematically

- Three distinct ideas relating color values to stimuli
  - Primaries and additive color: R, G, and B tell how much you turn up three *primary spectra*
  - Sensitivities and color detection: R, G, and B are the outputs of detectors with three *sensitivity functions*
  - Color matching functions and metamers: R, G, and B are the amounts of three primaries required to match a given spectrum

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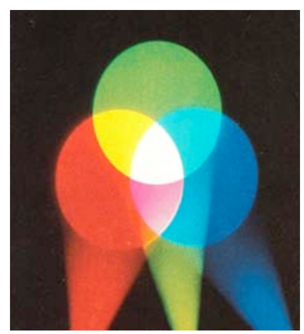
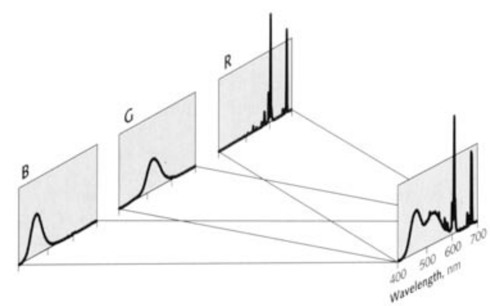
Math of additive mixing

- Simply add contributions of primaries per wavelength

$$s_a(\lambda) = R s_r(\lambda) + G s_g(\lambda) + B s_b(\lambda)$$

  - key property: all wavelengths change by the same scale factor





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A simple light detector

- Produces a number when photons land on it
  - value depends strictly on the number of photons detected
  - each photon has a probability of being detected that depends on the wavelength
  - (can't distinguish signals caused by different wavelengths)
- This model works for many detectors:
  - based on semiconductors (such as in a digital camera)
  - based on visual photopigments (such as in human eyes)

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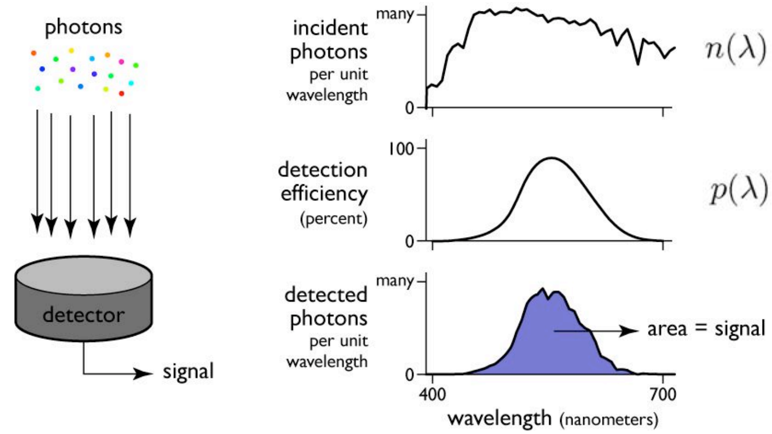
A simple light detector



$$X = \int p(\lambda) n(\lambda) \, d\lambda$$

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Light detection math

- Same math carries over to power distributions
  - spectrum entering the detector is $s(\lambda)$
  - detector has its *spectral sensitivity* or *spectral response*, $r(\lambda)$

$$X = \int r(\lambda) s(\lambda) \, d\lambda$$

measured signal

input spectrum

detector's sensitivity

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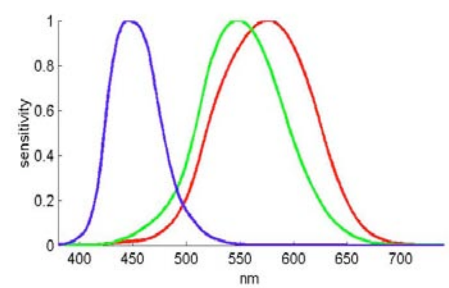
Light detection in the eye

- Recall there are three types of cones
  - call them S, M, L for short, medium, long wavelengths
  - eye therefore detects three values from a spectrum, corresponding to three response functions:

$$S = \int r_S(\lambda) s(\lambda) \, d\lambda$$

$$M = \int r_M(\lambda) s(\lambda) \, d\lambda$$

$$L = \int r_L(\lambda) s(\lambda) \, d\lambda$$



[Michael Murdoch | Kodak]

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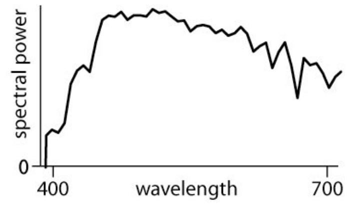
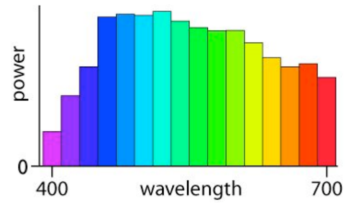
Spectra as vectors

- Additive synthesis and detection correspond to basic linear algebra concepts
  - for concreteness, think of spectra as having a finite number of little bands
  - continuous spectrum *s*(λ) becomes discrete spectrum *s*[*i*]



function $s(\lambda)$



vector $s[i]$

Color operations as vector algebra

- Additive display (synthesis):
  - linear combination of spectra:

$$s(\lambda) = R s_R(\lambda) + G s_G(\lambda) + B s_B(\lambda)$$

  - is like linear combination of vectors, or matrix multiplication:

$$s[i] = R s_R[i] + G s_G[i] + B s_B[i]$$

$$\begin{bmatrix} | \\ s \\ | \end{bmatrix} = \begin{bmatrix} | & | & | \\ s_R & s_G & s_B \\ | & | & | \end{bmatrix} \begin{bmatrix} R \\ G \\ B \end{bmatrix}$$

$$s = M_{RGB}\, C$$

Color operations as vector algebra

- Color detection (analysis):
  - linear measurement of spectra:

$$X = \int r(\lambda) s(\lambda)\, d\lambda$$

  - is like a dot product of vectors:

$$X = \sum_i r[i] s[i]$$

$$X = r \cdot s$$

Color operations as vector algebra

- Color detection (analysis):
  - three-band linear measurement of spectra corresponds to three dot products, or a matrix multiplication:

$$\begin{bmatrix} S \\ M \\ L \end{bmatrix} = \begin{bmatrix} r_S \cdot s \\ r_M \cdot s \\ r_L \cdot s \end{bmatrix} = \begin{bmatrix} — r_S — \\ — r_M — \\ — r_L — \end{bmatrix} \begin{bmatrix} | \\ s \\ | \end{bmatrix}$$

or,

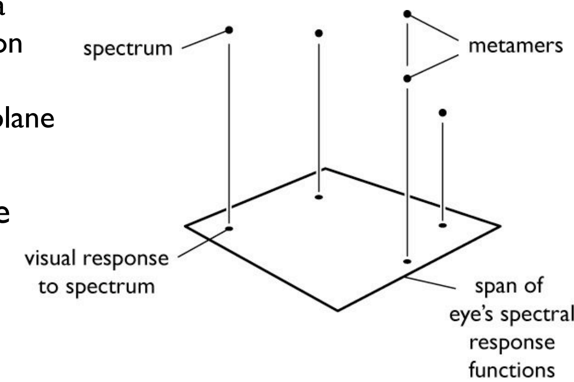
$$V = M_{SML}\, s.$$

Pseudo-geometric interpretation

- A dot product is a projection
- We are projecting a high dimensional vector (a spectrum) onto three vectors
  - differences that are perpendicular to all 3 vectors are not detectable
- For intuition, we can imagine a 3D analog
  - 3D stands in for high-D vectors
  - 2D stands in for 3D
  - Then vision is just projection onto a plane

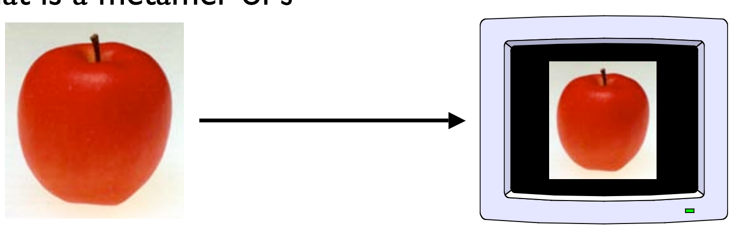
Pseudo-geometric interpretation

- The information available to the visual system about a spectrum is three values
  - this amounts to a loss of information analogous to projection on a plane
- Two spectra that produce the same response are metamers



Color reproduction

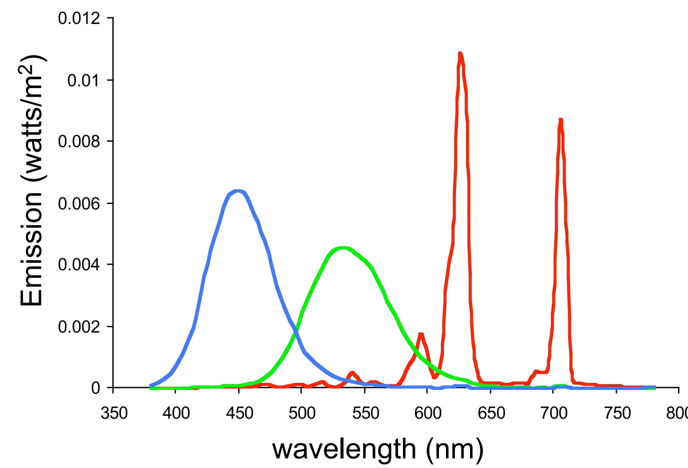
- Have a spectrum s; want to match on RGB monitor
  - "match" means it looks the same
  - any spectrum that projects to the same point in the visual color space is a good reproduction
- Must find a spectrum that the monitor *can* produce that is a metamer of s



R, G, B?

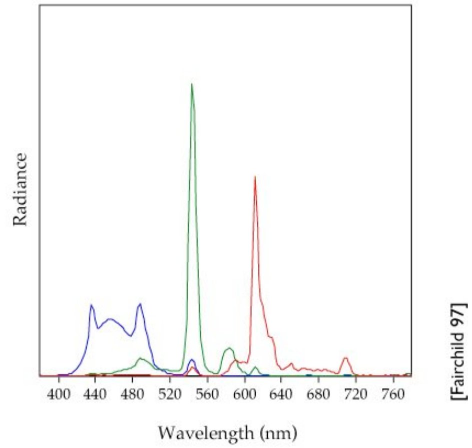
[cs417—Greenberg]

CRT display primaries



- Curves determined by phosphor emission properties

LCD display primaries



– Curves determined by (fluorescent) backlight and filters

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Color reproduction

- Say we have a spectrum *s* we want to match on an RGB monitor
  - "match" means it looks the same
  - any spectrum that projects to the same point in the visual color space is a good reproduction
- So, we want to find a spectrum s' that the monitor can produce that matches *s*
  - that is, we want to display a metamer of *s* on the screen

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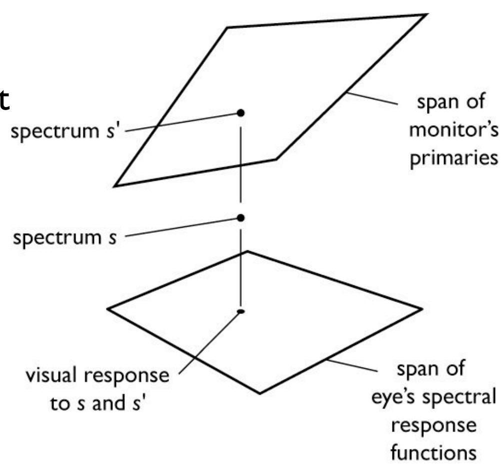
Color reproduction

- We want to compute the combination of r, g, b that will project to the same visual response as *s*.



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Color reproduction as linear algebra

- What color do we see when we look at the display?
  - Feed *C* to display
  - Display produces s'
  - Eye looks at s' and produces *V*

$$V = M_{SML} M_{RGB} C$$

$$\begin{bmatrix} S \\ M \\ L \end{bmatrix} = \begin{bmatrix} r_S \cdot s_R & r_S \cdot s_G & r_S \cdot s_B \\ r_M \cdot s_R & r_M \cdot s_G & r_M \cdot s_B \\ r_L \cdot s_R & r_L \cdot s_G & r_L \cdot s_B \end{bmatrix} \begin{bmatrix} R \\ G \\ B \end{bmatrix}$$

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Color reproduction as linear algebra

- Goal of reproduction: visual response to *s* and s_a is the same:

$$M_{SML}\, s = M_{SML}\, s'.$$

- Substituting in the expression for *s'*,

$$M_{SML}\, s = M_{SML} M_{RGB}\, C$$

$$C = \underbrace{(M_{SML} M_{RGB})^{-1} M_{SML}}_{\textit{color matching matrix}\text{ for RGB}}\, s$$

Color matching functions

- Used like response functions, but give primary weights
  - e.g. R,G,B color matching functions, dotted with a spectrum, tell how much of a particular R, G, and B are required to match the spectrum
- Just derived them for a particular display
  - also can measure directly
  - in fact, from visual experiments we can *only* get color matching functions, not S, M, and L
- Recall previous discussion: CIE XYZ system
  - standard hypothetical primaries defined only via color matching functions

Color matching in practice

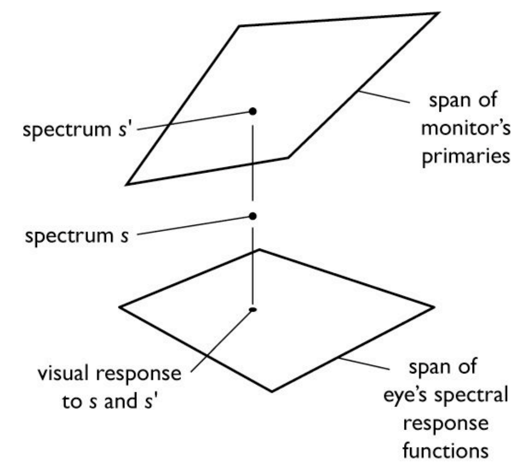
- In practice, we have color matching functions, not the S, M, and L sensitivities
  - but any color matching functions are just as good as SML for matching colors
  - any colors with the same X, Y, Z values have the same S, M, L values (they have to, because the colors match!)
  - so in practice color matching is done thus:

$$C = (M_{XYZ} M_{RGB})^{-1} M_{XYZ}\, s$$

    - and the results are the same as with M_{SML} because any color matching matrices span the same space

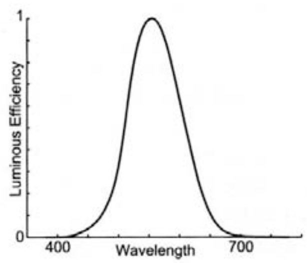
Color matching in practice

you can compute the point s' using any basis for the human visual subspace (you are just matching the response to s and s')



Basic colorimetric concepts

- Luminance
  - the overall magnitude of the the visual response to a spectrum (independent of its color)
    - corresponds to the everyday concept "brightness"
  - determined by product of SPD with the *luminous efficiency function* V_λ that describes the eye's overall ability to detect light at each wavelength
  - e.g. lamps are optimized to improve their luminous efficiency (tungsten vs. fluorescent vs. sodium vapor)



[Stone 2003]

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Luminance, mathematically

- Y just has another response curve (like *S*, *M*, and *L*)

$$Y = r_Y \cdot s$$

  - r_Y is really called "V_λ"
- V_λ is a linear combination of *S*, *M*, and *L*
  - Has to be, since it's derived from cone outputs

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Color spaces

- Need three numbers to specify a color
  - but what three numbers?
  - a *color space* is an answer to this question
- Common example: monitor RGB
  - define colors by what R, G, B signals will produce them on your monitor
    (in math, s = *R***R** + *G***G** + *B***B** for some spectra **R**, **G**, **B**)
  - device dependent (depends on gamma, phosphors, gains, …)
    - therefore if I choose RGB by looking at my monitor and send it to you, you may not see the same color
  - also leaves out some colors (limited *gamut*), e.g. vivid yellow

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Standard color spaces

- Standardized RGB (sRGB)
  - makes a particular monitor RGB standard
  - other color devices simulate that monitor by calibration
  - sRGB is usable as an interchange space; widely adopted today
  - gamut is still limited

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A universal color space: XYZ

- Standardized by CIE (*Commission Internationale de l'Eclairage,* the standards organization for color science)
- Based on three "imaginary" primaries **X**, **Y**, and **Z**

  (in math, s = $X\mathbf{X} + Y\mathbf{Y} + Z\mathbf{Z}$)
  - imaginary = only realizable by spectra that are negative at some wavelengths
  - key properties
    - *any* stimulus can be matched with positive *X*, *Y*, and *Z*
    - separates out luminance: **X**, **Z** have zero luminance, so *Y* tells you the luminance by itself

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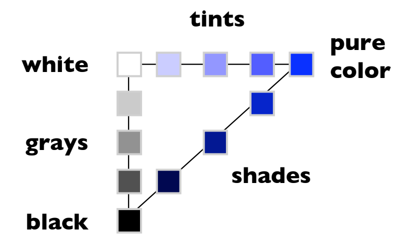
Perceptually organized color spaces

- Artists often refer to colors as *tints*, *shades*, and *tones* of pure pigments
  - tint: mixture with white
  - shade: mixture with black
  - tones: mixture with black and white
  - gray: no color at all (aka. neutral)



[after FvDFH]

- This seems intuitive
  - tints and shades are inherently related to the pure color
    - "same" color but lighter, darker, paler, etc.

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Perceptual dimensions of color

- Hue
  - the "kind" of color, regardless of attributes
  - colorimetric correlate: dominant wavelength
  - artist's correlate: the chosen pigment color
- Saturation
  - the "colorfulness"
  - colorimetric correlate: purity
  - artist's correlate: fraction of paint from the colored tube
- Lightness (or value)
  - the overall amount of light
  - colorimetric correlate: luminance
  - artist's correlate: tints are lighter, shades are darker

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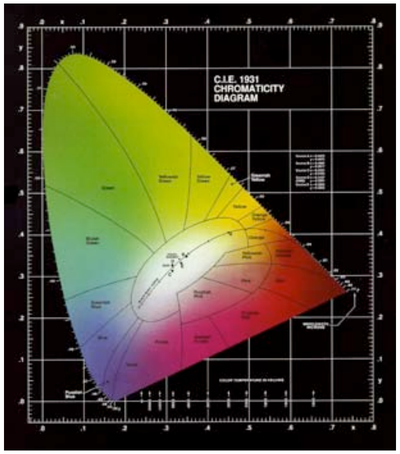
Perceptual dimensions: chromaticity

- In x, y, Y (or another luminance/chromaticity space), Y corresponds to lightness
- hue and saturation are then like polar coordinates for chromaticity (starting at white, which way did you go and how far?)



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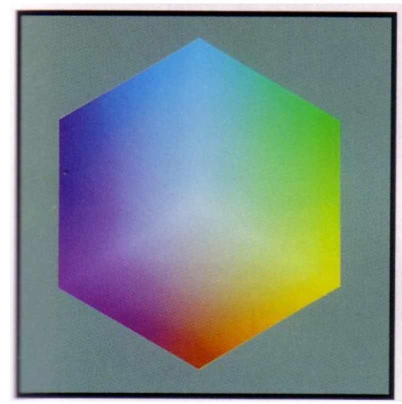
Perceptual dimensions of color

- There's good evidence ("opponent color theory") for a neurological basis for these dimensions
  - the brain seems to encode color early on using three axes:
    white — black, red — green, yellow — blue
  - the white—black axis is lightness; the others determine hue and saturation
  - one piece of evidence: you can have a light green, a dark green, a yellow-green, or a blue-green, but you can't have a reddish green (just doesn't make sense)
    - thus red is the *opponent* to green
  - another piece of evidence: afterimages (recall flag illusion)

RGB as a 3D space

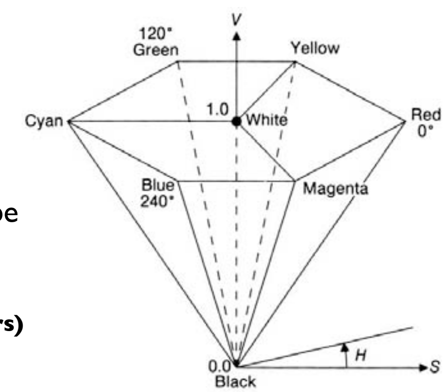
- A cube:



(demo of RGB color picker)

Perceptual organization for RGB: HSV

- Uses hue (an angle, 0 to 360), saturation (0 to 1), and value (0 to 1) as the three coordinates for a color
  - the brightest available RGB colors are those with one of R,G,B equal to 1 (top surface)
  - each horizontal slice is the surface of a sub-cube of the RGB cube



[FvDFH]

(demo of HSV color pickers)

Perceptually uniform spaces

- Two major spaces standardized by CIE
  - designed so that equal differences in coordinates produce equally visible differences in color
  - LUV: earlier, simpler space; *L**, *u**, *v**
  - LAB: more complex but more uniform: *L**, *a**, *b**
  - both separate luminance from chromaticity
  - including a gamma-like nonlinear component is important