Numerical analysis: Homework 4

Instructor: Anil Damle
Due: April 12, 2024

Policies

You may discuss the homework problems freely with other students, but please refrain from looking at their code or writeups (or sharing your own). Ultimately, you must implement your own code and write up your own solution to be turned in. Your solution, including plots and requested output from your code should be typeset and submitted via the Gradescope as a pdf file. This file must be self contained for grading. Additionally, please submit any code written for the assignment as zip file to the separate Gradescope assignment for code.

Question 1:

Implement Newton's method for root finding. For each of the following compute a root of the function and illustrate the order of convergence (i.e., q if $|e_{k+1}| = \rho |e_k|^q$, where $e_k = x^* - x_k$) and, if linear, the rate (i.e., ρ if $|e_{k+1}| = \rho |e_k|$) exhibited by the method. Discuss if you observe what you expect.

(a) $f(x) = x^2$

(b) $f(x) = \sin x + x^3$

(c) $f(x) = \sin \frac{1}{x}$ for $x \neq 0$

Question 2:

Show that given any initial guess the Jacobi method for solving $Ax = b$converges for any strictly diagonally dominant matrix A (i.e., $|a_{ii}| > \sum_{j \neq i} |a_{ij}|$ for $i = 1, \ldots, n$).

You may find the following (restricted) version of the Gershgorin circle theorem useful for this question: Given a matrix A, each of its eigenvalues lies within at least one of the discs $\{z \in \mathbb{C} \mid |z - a_{ii}| < R_i\}$ for $i = 1, \ldots, n$, where $R_i = \sum_{j \neq i} |a_{ij}|$. In other words, all of the eigenvalues of A lie within the union of a set of discs centered around the diagonal entries of A, each of whose radius is the sum of the magnitudes of the off diagonal entries in that row.

Question 3:

Prove that if $\nabla f(x) = 0$ but the Hessian $\nabla^2 f(x)$ is indefinite (i.e., has positive and negative eigenvalues), then there is a direction we can move from x that decreases the function value. (I.e., show that x is not a local minimizer and that we can make progress when running an optimization scheme using so-called directions of negative curvature.)

Question 4 (an interesting ungraded problem):

Suppose we have a function $f : \mathbb{R}^n \to \mathbb{R}$ with a local minimizer x^* such that for any direction $p \in \mathbb{R}^n$ (say with $\|p\|_2 = 1$) there exists an $\epsilon > 0$ such that $f(x^* + \alpha p) > f(x^*)$ for all $\alpha \in (-\epsilon, \epsilon)$. Does this guarantee that x^* is a strict local minimizer of $f(x)$? Why or why not?